An Illustration. The differential equation $y'' - y = x + xe^x$ will be solved, verifying that $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$.

Solution:

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ has roots $r = \pm 1$. Recipe case 1 implies $y_h = c_1e^x + c_2e^{-x}$.

Initial trial solution. The atoms of $f(x) = x + xe^x$ are $x$ and $xe^x$. Differentiation of these atoms gives a new list of four distinct atoms $1, x, e^x, xe^x$. The undetermined coefficients will be assigned the symbols $d_1, d_2, d_3, d_4$. Then the initial trial solution is

$$y = d_1 + d_2x + d_3e^x + d_4xe^x.$$

Fixup rule. The atoms of $y_h = c_1e^x + c_2e^{-x}$ form the list $L = \{e^x, e^{-x}\}$. In $y$, term $d_3e^x$ contains atom $e^x$ from list $L$. Term $d_3e^x$ and related atom term $d_4xe^x$ are replaced by $x(d_3e^x)$ and $x(d_4xe^x)$, respectively. Terms $d_1$ and $d_2x$ are unaffected by the fixup rule. The corrected trial solution is

$$y = d_1 + d_2x + d_3xe^x + d_4x^2e^x.$$

Because no term of $y$ contains an atom in list $L$, this is the final trial solution.

Substitute $y$ into $y'' - y = x + xe^x$. The details:
LHS = $y'' - y$

Left side of the equation.

$$= [y''_1 - y_1] + [y''_2 - y_2]$$

Let $y = y_1 + y_2$, $y_1 = d_1 + d_2x$, $y_2 = d_3xe^x + d_4x^2e^x$.

$$= [0 - y_1] + [2d_3e^x + 2d_4e^x + 4d_4xe^x]$$

Use $y''_1 = 0$ and $y''_2 = y_2 + 2d_3e^x + 2d_4e^x + 4d_4xe^x$.

$$= (-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x$$

Collect on distinct atoms.

Write out a $4 \times 4$ system. Because LHS = RHS and RHS = $x + xe^x$, the last display gives the relation

$$(-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x = x + xe^x. \quad (2)$$

Equate coefficients of matching atoms left and right to give the system of equations

$$-d_1 = 0, \quad -d_2 = 1, \quad 2d_3 + 2d_4 = 0, \quad 4d_4 = 1. \quad (3)$$

Atom matching effectively removes $x$ and changes the equation into a $4 \times 4$ linear system for symbols $d_1, d_2, d_3, d_4$. 
The technique is independence. To explain, independence of atoms means that a linear combination of atoms is uniquely represented, hence two such equal representations must have matching coefficients. Relation (2) says that two linear combinations of the same list of atoms are equal. Hence coefficients left and right in (2) must match, which gives $4 \times 4$ system (3).

**Solve the equations.** The $4 \times 4$ system must always have a unique solution. Equivalently, there are four lead variables and zero free variables. Solving by back-substitution gives $d_1 = 0$, $d_2 = -1$, $d_4 = 1/4$, $d_3 = -1/4$.

**Report $y_p$.** The trial solution with determined coefficients $d_1 = 0$, $d_2 = -1$, $d_3 = -1/4$, $d_4 = 1/4$ becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$ 

**Report $y = y_h + y_p$.** From above, $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$. Then $y = y_h + y_p$ is given by

$$y = c_1e^x + c_2e^{-x} - x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$ 

**Answer check.** Computer algebra system *maple* is used.

```maple
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success is a report of zero.
```