An Illustration. The differential equation $y'' - y = x + xe^x$ will be solved, verifying that $y_h = c_1 e^x + c_2 e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2 e^x$.

Solution:

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ has roots $r = \pm 1$. Recipe **case 1** implies $y_h = c_1 e^x + c_2 e^{-x}$.

Initial trial solution. The atoms of $f(x) = x + xe^x$ are x and xe^x. Differentiation of these atoms gives a new list of four distinct atoms 1, x, e^x, xe^x. The undetermined coefficients will be assigned the symbols d_1, d_2, d_3, d_4. Then the initial trial solution is

$$y = d_1 + d_2 x + d_3 e^x + d_4 x e^x.$$

Fixup rule. The atoms of $y_h = c_1 e^x + c_2 e^{-x}$ form the list $L = \{e^x, e^{-x}\}$. In y, term $d_3 e^x$ contains atom e^x from list L. Term $d_3 e^x$ and **related atom** term $d_4 x e^x$ are replaced by $x(d_3 e^x)$ and $x(d_4 x e^x)$, respectively. Terms d_1 and $d_2 x$ are unaffected by the fixup rule. The corrected trial solution is

$$y = d_1 + d_2 x + d_3 x e^x + d_4 x^2 e^x.$$

Because no term of y contains an atom in list L, this is the final trial solution.

Substitute y into $y'' - y = x + xe^x$. The details:

$\text{LHS} = y'' - y$	Left side of the equation.
$= [y_1'' - y_1] + [y_2'' - y_2]$	Let $y = y_1 + y_2$, $y_1 = d_1 + d_2 x$, $y_2 = d_3 x e^x + d_4 x^2 e^x$.
$= [0 - y_1] + [2d_3 e^x + 2d_4 e^x + 4d_4 x e^x]$	Use $y_1'' = 0$ and $y_2'' = y_2 + 2d_3 e^x + 2d_4 e^x + 4d_4 x e^x$.
$= (-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x$	Collect on distinct atoms.

Write out a 4×4 **system**. Because LHS = RHS and $\text{RHS} = x + xe^x$, the last display gives the relation

$$\begin{aligned}(-d_1)1 + (-d_2)x + \\ (2d_3 + 2d_4)e^x + (4d_4)xe^x &= x + xe^x.\end{aligned} \tag{2}$$

Equate coefficients of matching atoms left and right to give the system of equations

$$\begin{array}{rrrrcl} -d_1 & & & & = & 0, \\ & -d_2 & & & = & 1, \\ & & 2d_3 & +2d_4 & = & 0, \\ & & & 4d_4 & = & 1. \end{array} \tag{3}$$

Atom matching effectively removes x and changes the equation into a 4×4 linear system for symbols d_1, d_2, d_3, d_4.

The technique is independence. To explain, independence of atoms means that a linear combination of atoms is uniquely represented, hence two such equal representations must have matching coefficients. Relation (2) says that two linear combinations of the same list of atoms are equal. Hence coefficients left and right in (2) must match, which gives 4×4 system (3).

Solve the equations. The 4×4 system must always have a unique solution. Equivalently, there are four lead variables and zero free variables. Solving by back-substitution gives $d_1 = 0$, $d_2 = -1$, $d_4 = 1/4$, $d_3 = -1/4$.

Report y_p. The trial solution with determined coefficients $d_1 = 0$, $d_2 = -1$, $d_3 = -1/4$, $d_4 = 1/4$ becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Report $y = y_h + y_p$. From above, $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$. Then $y = y_h + y_p$ is given by

$$y = c_1e^x + c^2e^{-x} - x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Answer check. Computer algebra system `maple` is used.

```
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success is a report of zero.
```