2.4 Linear Equations II

Studied here are the subjects of variation of parameters and undetermined coefficients for linear first order differential equations.

Variation of Parameters

A particular solution $y_p^*(x)$ of the non-homogeneous equation

$$(1) \qquad y' + p(x)y = r(x)$$

is given by equation (5) in 2.3. Literature calls it the **variation of parameters** formula or the **variation of constants** formula.

Theorem 3 (Variation of Parameters)
A particular solution of the differential equation $y' + p(x)y = r(x)$ is given by either of the formulae

$$(2) \qquad y_p^*(x) = e^{-\int_{x_0}^x p(s)ds} \int_{x_0}^x r(t) e^{\int_{x_0}^t p(s)ds}\, dt,$$

$$(3) \qquad y_p(x) = e^{-\int p(x)dx} \int r(x) e^{\int p(x)dx}\, dx.$$

Indefinite Integrals. The indefinite integral form (3) is used in science and engineering applications. The answers (2) and (3) differ by a solution of the homogeneous equation: $y_p^*(x) = y_p(x) + y_h(x)$ for some choice of the constant c in y_h. Both answers (2) and (3) are solutions of the nonhomogeneous differential equation, even though (2) generally contains an extra term. While (2) satisfies $y(x_0) = 0$, (3) may not.

Integrating Factor Formula. An integrating factor for (1) is

$$Q(x) = e^{\int p(x)dx}.$$

Formula (3) can be written in terms of $Q(x)$ as

$$y_p(x) = \frac{1}{Q(x)} \int r(x)Q(x)dx.$$

Compact Formula. Because $\int_x^t f = \int_x^{x_0} f + \int_{x_0}^t f$ and $\int_x^{x_0} f = -\int_{x_0}^x f$, the exponential factors in (2) can be re-written as

$$(4) \qquad y_p(x) = \int_{x_0}^x r(t) e^{\int_x^t p(s)ds}\, dt.$$

The reader is warned that using indefinite integrals in (4) results in the wrong answer.

Terminology. The name *variation of parameters* comes from the idea of *varying the parameter* c in the homogeneous solution formula $y_h = c\mathbf{R}(x)$, where $\mathbf{R}(x) = e^{-\int p(x)dx}$. Historically, c is replaced by an unknown function $y_0(x)$, to define a *trial solution* $y(x) = y_0(x)\mathbf{R}(x)$ of (1). A derivation appears on page 93.

The Method of Undetermined Coefficients

The method applies to $y' + p(x)y = r(x)$. It finds a particular solution y_p *without* the integration steps present in variation of parameters. The requirements and limitations:

1. Coefficient $p(x)$ of $y' + p(x)y = r(x)$ is constant.
2. The function $r(x)$ is a sum of constants times atoms.

An **atom** is a term having one of the forms

$$x^m,\ x^m e^{ax},\ x^m \cos bx,\ x^m \sin bx,\ x^m e^{ax} \cos bx \quad \text{or} \quad x^m e^{ax} \sin bx.$$

The symbols a and b are real constants, with $b > 0$. Symbol $m \geq 0$ is an integer. The terms x^3, $x \cos 2x$, $\sin x$, e^{-x}, $x^6 e^{-\pi x}$ are atoms. Conversely, if $r(x) = 4 \sin x + 5xe^x$, then split the sum into terms and drop the coefficients 4 and 5 to identify atoms $\sin x$ and xe^x.

The Method.

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** $d_1, \ldots, d_k$, then add to define a **trial solution** y.
2. **Fixup rule**: if solution e^{-px} of $y' + py = 0$ appears in trial solution y, then replace in y matching atoms e^{-px}, xe^{-px}, ... by xe^{-px}, x^2e^{-px}, ... (other atoms appearing in y are unchanged). The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y' + py = r(x)$. Match coefficients of atoms left and right to write out linear algebraic equations for the undetermined coefficients $d_1, \ldots, d_k$.
4. Solve the equations. The trial solution y with evaluated coefficients $d_1, \ldots, d_k$ becomes the particular solution y_p.

Undetermined Coefficients Illustrated.

We will solve

$$y' + 2y = xe^x + 2x + 1 + 3 \sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3 \sin x$ is a sum of terms constructed from the atoms xe^x, x, 1, $\sin x$. The left side is $y' + p(x)y$ with $p(x) = 2$, a constant. Therefore, the method of undetermined coefficients applies to find y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (drop coefficients to identify new atoms). The solution e^{-2x} of $y' + 2y = 0$ does not appear in the list of atoms, so the fixup rule does not apply. Then the trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

Equations. To substitute the trial solution y into $y' + 2y$ requires a formula for y':

$$y' = d_2 + d_3 e^x + d_4 x e^x + d_4 e^x - d_5 \sin x + d_6 \cos x.$$

Then

$$\begin{aligned} r(x) &= y' + 2y \\ &= d_2 + d_3 e^x + d_4 x e^x + d_4 e^x - d_5 \sin x + d_6 \cos x \\ &\quad + 2d_1 + 2d_2 x + 2d_3 e^x + 2d_4 x e^x + 2d_5 \cos x + 2d_6 \sin x \\ &= (d_2 + 2d_1)(1) + 2d_2(x) + (3d_3 + d_4)(e^x) + (3d_4)(xe^x) \\ &\quad + (2d_5 + d_6)(\cos x) + (2d_6 - d_5)(\sin x) \end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. For instance, constant term $d_2 + 2d_1$ in the expansion of $y' + 2y$ matches constant term 1 in $r(x)$. Writing out the matches gives the equations

$$\begin{aligned} 2d_1 + d_2 &= 1, \\ 2d_2 &= 2, \\ 3d_3 + d_4 &= 0, \\ 3d_4 &= 1, \\ 2d_5 + d_6 &= 0, \\ -d_5 + 2d_6 &= 3. \end{aligned}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = 1$, $d_1 = 0$, $d_4 = 1/3$, $d_3 = -1/9$. The last two equations are solved by elimination or Cramer's rule (reviewed in Chapter 3) to give $d_6 = 6/5$, $d_5 = -3/5$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = x - \frac{1}{9}e^x + \frac{1}{3}xe^x - \frac{3}{5}\cos x + \frac{6}{5}\sin x.$$

A Fixup Rule Illustration.

Solve the equation

$$y' + 3y = 8e^x + 3x^2 e^{-3x}$$

by the method of undetermined coefficients. Verify that the general solution $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = 2e^x + x^3 e^{-3x}.$$

Solution: The right side $r(x) = 8e^x + 3x^2 e^{-3x}$ is constructed from atoms e^x, $x^2 e^{-3x}$. Repeated differentiation of these atoms identifies the new list of atoms e^x, e^{-3x}, xe^{-3x}, $x^2 e^{-3x}$. The fixup rule applies because the solution e^{-3x} of

$y' + 3y = 0$ appears in the list. The atoms of the form $x^m e^{-3x}$ are multiplied by x to give the new list of atoms e^x, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}. Readers should take note that atom e^x is unaffected by the fixup rule modification. Then the corrected trial solution is

$$y = d_1e^x + d_2xe^{-3x} + d_3x^2e^{-3x} + d_4x^3e^{-3x}.$$

The trial solution expression y is substituted into $y' + 3y = 2e^x + x^2e^{-3x}$ to give the equation

$$4d_1e^x + d_2e^{-3x} + 2d_3xe^{-3x} + 3d_4x^2e^{-3x} = 8e^x + 3x^2e^{-3x}.$$

Coefficients of atoms on each side of the preceding equation are matched to give the equations

$$\begin{aligned} 4d_1 \qquad\qquad\qquad &= 8, \\ d_2 \qquad\qquad &= 0, \\ 2d_3 \qquad &= 0, \\ 3d_4 &= 3. \end{aligned}$$

Then $d_1 = 2$, $d_2 = d_3 = 0$, $d_4 = 1$ and the particular solution is reported to be $y_p = 2e^x + x^3e^{-3x}$.

Examples

16 **Example (Variation of Parameters Method)** Solve the equation $2y' + 6y = 4xe^{-3x}$ by the method of variation of parameters, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = x^2e^{-3x}.$$

Solution: Divide the equation by 2 to obtain the standard linear form

$$y' + 3y = 2xe^{-3x}.$$

Solution y_h. The homogeneous equation $y' + 3y = 0$ is solved by the *recipe* to give $y_h = ce^{-3x}$.

Solution y_p. Identify $p(x) = 3$, $r(x) = 2xe^{-3x}$ from the standard form. The mechanics: let $y' = f(x, y) \equiv 2xe^{-3x} - 3y$ and define $r(x) = f(x, 0)$, $p(x) = -f_y(x, y) = 3$. The variation of parameters formula is applied as follows. First, compute the integrating factor $Q(x) = e^{\int p(x)dx} = e^{3x}$. Then

$$\begin{aligned} y_p(x) &= (1/Q(x)) \int r(x)Q(x)dx \\ &= e^{-3x} \textstyle\int 2xe^{-3x}e^{3x}dx \\ &= x^2e^{-3x}. \end{aligned}$$

It must be explained that all integration constants were set to zero, in order to obtain the shortest possible expression for y_p. Indeed, if $Q = e^{3x+c_1}$ instead of e^{3x}, then the factors $1/Q$ and Q contribute constant factors $1/e^{c_1}$ and e^{c_1}, which multiply to one; the effect is to set $c_1 = 0$. On the other hand, an integration constant c_2 added to $\int r(x)Q(x)dx$ adds the homogeneous solution c_2e^{-3x} to the expression for y_p. Because we seek the shortest expression which is a solution to the non-homogeneous differential equation, the constant c_2 is set to zero.

17 Example (Undetermined Coefficient Method) Solve the equation $2y' + 6y = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ by the method of undetermined coefficients, verifying $y = y_h + y_p$ is given by

$$y_h = ce^{-3x}, \quad y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Solution: The method applies, because the differential equation $2y' + 6y = 0$ has constant coefficients and the right side $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$ is constructed from the list of atoms xe^{-x}, xe^{-3x}, $\sin x$.

List of Atoms. Differentiate the atoms xe^{-x}, xe^{-3x}, $\sin x$ to find the new list of atoms e^{-x}, xe^{-x}, e^{-3x}, xe^{-3x}, $\cos x$, $\sin x$. The solution e^{-3x} of $2y' + 6y = 0$ appears in the list: the fixup rule applies. Then e^{-3x}, xe^{-3x} are replaced by xe^{-3x}, x^2e^{-3x} to give the corrected list of atoms e^{-x}, xe^{-x}, xe^{-3x}, x^2e^{-3x}, $\cos x$, $\sin x$. Please note that only two of the six atoms were corrected.

Trial solution. The corrected trial solution is

$$y = d_1e^{-x} + d_2xe^{-x} + d_3xe^{-3x} + d_4x^2e^{-3x} + d_5\cos x + d_6\sin x.$$

Substitute y into $2y' + 6y = r(x)$ to give

$$\begin{aligned} r(x) &= 2y' + 6y \\ &= (4d_1 + 2d_2)e^{-x} + 4d_2xe^{-x} + 2d_3e^{-3x} + 4d_4xe^{-3x} \\ &\quad +(2d_6 + 6d_5)\cos x + (6d_6 - 2d_5)\sin x. \end{aligned}$$

Equations. Matching atoms on the left and right of $2y' + 6y = r(x)$, given $r(x) = 4xe^{-x} + 4xe^{-3x} + 5\sin x$, justifies the following equations for the undetermined coefficients; the solution is $d_2 = 1$, $d_1 = -1/2$, $d_3 = 0$, $d_4 = 1$, $d_6 = 3/4$, $d_5 = -1/4$.

$$\begin{aligned} 4d_1 + 2d_2 &= 0, \\ 4d_2 &= 4, \\ 2d_3 &= 0, \\ 4d_4 &= 4, \\ 6d_5 + 2d_6 &= 0, \\ -2d_5 + 6d_6 &= 5. \end{aligned}$$

Report. The trial solution upon substitution of the values for the undetermined coefficients becomes

$$y_p = -\frac{1}{2}e^{-x} + xe^{-x} + x^2e^{-3x} - \frac{1}{4}\cos x + \frac{3}{4}\sin x.$$

Details

Historical Account of Variation of Parameters.

$r = y' + py$	Let $\mathbf{R}(x) = e^{-\int p(x)dx}$. Assume $y = y_0(x)\mathbf{R}(x)$ solves $y' + py = r$.
$= (y_0\mathbf{R})' + py_0\mathbf{R}$	Substitute $y = y_0(x)\mathbf{R}(x)$ but suppress x.

$$= y_0'\mathbf{R} + y_0\mathbf{R}' + py_0\mathbf{R}$$ Apply the product rule $(uv)' = u'v + uv'$.

$$= y_0'\mathbf{R} - y_0p\mathbf{R} + py_0\mathbf{R}$$ Let $\mathcal{Q} = e^{\int p(x)dx}$. Apply $\mathcal{Q}' = -p\mathcal{Q}$.

$$= y_0'/\mathcal{Q}$$ Because $1/\mathbf{R} = \mathcal{Q}$.

The calculation gives $y_0'(x) = r(x)\mathcal{Q}(x)$. The method of quadrature applies to determine $y_0(x) = \int_{x_0}^{x}(r(t)\mathcal{Q}(t))dt$, because $y_0 = 0$ at $x = x_0$. Then $y = y_0\mathbf{R}$ duplicates the formula for y_p^* given in (5), which is equivalent to (2).

Exercises 2.4

Variation of Parameters I. Report the shortest particular solution given by the formula

$$y_p(x) = \frac{\int rQ}{Q}, \quad Q = e^{\int p}.$$

1. $y' = x + 1$

2. $y' = 2x - 1$

3. $y' + y = e^{-x}$

4. $y' + y = e^{-2x}$

5. $y' - 2y = 1$

6. $y' - y = 1$

7. $2y' + y = e^{x}$

8. $2y' + y = e^{-x}$

9. $xy' = x + 1$

10. $xy' = 1 - x^2$

Variation of Parameters II. Compute the particular solution given by

$$y_p^*(x) = \frac{\int_{x_0}^{x} rQ}{Q(t)}, \quad Q(t) = e^{\int_{x_0}^{t} p}.$$

11. $y' = x + 1$, $x_0 = 0$

12. $y' = 2x - 1$, $x_0 = 0$

13. $y' + y = e^{-x}$, $x_0 = 0$

14. $y' + y = e^{-2x}$, $x_0 = 0$

15. $y' - 2y = 1$, $x_0 = 0$

16. $y' - y = 1$, $x_0 = 0$

17. $2y' + y = e^{x}$, $x_0 = 1$

18. $2y' + y = e^{-x}$, $x_0 = 1$

19. $xy' = x + 1$, $x_0 = 1$

20. $xy' = 1 - x^2$, $x_0 = 1$

Atoms. Report the list of distinct atoms of the given function $f(x)$.

21. $x + e^{x}$

22. $1 + 2x + 5e^{x}$

23. $x(1 + x + 2e^{x})$

24. $x^2(2 + x^2) + x^2e^{-x}$

25. $\sin x \cos x + e^{x} \sin 2x$

26. $\cos^2 x - \sin^2 x + x^2e^{x} \cos 2x$

27. $(1 + 2x + 4x^5)e^{x}e^{-3x}e^{x/2}$

28. $(1+2x+4x^5+e^{x} \sin 2x)e^{-3x/4}e^{x/2}$

29. $\dfrac{x + e^{x}}{e^{-2x}} \sin 3x + e^{3x} \cos 3x$

30. $\dfrac{x + e^{x} \sin 2x + x^3}{e^{-2x}} \sin 5x$

Initial Trial Solution. Differentiate repeatedly $f(x)$ and report the list of distinct atoms which appear in f and all its derivatives.

31. $12 + 5x^2 + 6x^7$

32. $x^6/x^{-4} + 10x^4/x^{-6}$

33. $x^2 + e^{x}$

34. $x^3 + 5e^{2x}$

35. $(1 + x + x^3)e^{x} + \cos 2x$

36. $(x+e^x)\sin x + (x-e^{-x})\cos 2x$

37. $(x+e^x+\sin 3x+\cos 2x)e^{-2x}$

38. $(x^2e^{-x}+4\cos 3x+5\sin 2x)e^{-3x}$

39. $(1+x^2)(\sin x\cos x-\sin 2x)e^{-x}$

40. $(8-x^3)(\cos^2 x-\sin^2 x)e^{3x}$

Fixup Rule. Given the homogeneous solution y_h and an initial trial solution y, determine the final trial solution according to the fixup rule.

41. $y_h(x)=ce^{2x}$, $y=d_1+d_2x+d_3e^{2x}$

42. $y_h(x)=ce^{2x}$, $y=d_1+d_2e^{2x}+d_3xe^{2x}$

43. $y_h(x)=ce^{0x}$, $y=d_1+d_2x+d_3x^2$

44. $y_h(x)=ce^{x}$, $y=d_1+d_2x+d_3x^2$

45. $y_h(x)=ce^{x}$, $y=d_1\cos x+d_2\sin x+d_3e^x$

46. $y_h(x)=ce^{2x}$, $y=d_1e^{2x}\cos x+d_2e^{2x}\sin x$

47. $y_h(x)=ce^{2x}$, $y=d_1e^{2x}+d_2xe^{2x}+d_3x^2e^{2x}$

48. $y_h(x)=ce^{-2x}$, $y=d_1e^{-2x}+d_2xe^{-2x}+d_3e^{2x}+d_4xe^{2x}$

49. $y_h(x)=cx^2$, $y=d_1+d_2x+d_3x^2$

50. $y_h(x)=cx^3$, $y=d_1+d_2x+d_3x^2$

Undetermined Coefficients: Trial Solution. Find the form of the **corrected** trial solution y but do not evaluate the undetermined coefficients.

51. $y'=x^3+5+x^2e^x(3+2x+\sin 2x)$

52. $y'=x^2+5x+2+x^3e^x(2+3x+5\cos 4x)$

53. $y'-y=x^3+2x+5+x^4e^x(2+4x+7\cos 2x)$

54. $y'-y=x^4+5x+2+x^3e^x(2+3x+5\cos 4x)$

55. $y'-2y=x^3+x^2+x^3e^x(2e^x+3x+5\sin 4x)$

56. $y'-2y=x^3e^{2x}+x^2e^x(3+4e^x+2\cos 2x)$

57. $y'+y=x^2+5x+2+x^3e^{-x}(6x+3\sin x+2\cos x)$

58. $y'-2y=x^5+5x^3+14+x^3e^x(5+7xe^{-3x})$

59. $2y'+4y=x^4+5x^5+2x^8+x^3e^x(7+5xe^x+5\sin 11x)$

60. $5y'+y=x^2+5x+2e^{x/5}+x^3e^{x/5}(7+9x+2\sin(9x/2))$

Undetermined Coefficients. Compute a particular solution y_p according to the method of undetermined coefficients. Report (1) the initial trial solution, (2) the corrected trial solution, (3) the system of equations for the undetermined coefficients and finally (4) the formula for y_p.

61. $y'+y=x+1$

62. $y'+y=2x-1$

63. $y'-y=e^x+e^{-x}$

64. $y'-y=xe^x+e^{-x}$

65. $y'-2y=1+x+e^{2x}+\sin x$

66. $y'-2y=1+x+xe^{2x}+\cos x$

67. $y'+2y=xe^{-2x}+x^3$

68. $y'+2y=(2+x)e^{-2x}+xe^x$

69. $y'=x^2+4+xe^x(3+\cos x)$

70. $y'=x^2+5+xe^x(2+\sin x)$