Chapter 7

Laplace Transform

The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, the **Laplace method** is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive.

The **direct Laplace transform** or the **Laplace integral** of a function $f(t)$ defined for $0 \le t < \infty$ is the ordinary calculus integration problem

$$\int_0^\infty f(t)e^{-st}dt,$$

succinctly denoted $\mathcal{L}(f(t))$ in science and engineering literature. The $\mathcal{L}$–notation recognizes that integration always proceeds over $t = 0$ to $t = \infty$ and that the integral involves an *integrator* $e^{-st}dt$ instead of the usual dt. These minor differences distinguish **Laplace integrals** from the ordinary integrals found on the inside covers of calculus texts.

7.1 Introduction to the Laplace Method

The foundation of Laplace theory is **Lerch's cancellation law**

(1)
$$\begin{array}{lcl} \int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt & \text{implies} & y(t) = f(t), \\ & \text{or} & \\ \mathcal{L}(y(t) = \mathcal{L}(f(t)) & \text{implies} & y(t) = f(t). \end{array}$$

In differential equation applications, $y(t)$ is the sought-after unknown while $f(t)$ is an explicit expression taken from integral tables.

Below, we illustrate Laplace's method by solving the initial value problem

$$y' = -1, \quad y(0) = 0.$$

The method obtains a relation $\mathcal{L}(y(t)) = \mathcal{L}(-t)$, whence Lerch's cancellation law implies the solution is $y(t) = -t$.

The **Laplace method** is advertised as a *table lookup method*, in which the solution $y(t)$ to a differential equation is found by looking up the answer in a special integral table.

Laplace Integral. The integral $\int_0^\infty g(t)e^{-st}dt$ is called the **Laplace integral** of the function $g(t)$. It is defined by $\lim_{N\to\infty}\int_0^N g(t)e^{-st}dt$ and depends on variable s. The ideas will be illustrated for $g(t)=1$, $g(t)=t$ and $g(t)=t^2$, producing the integral formulas in Table 1.

$\int_0^\infty (1)e^{-st}dt = -(1/s)e^{-st}\big\rvert_{t=0}^{t=\infty}$	Laplace integral of $g(t)=1$.
$= 1/s$	Assumed $s>0$.
$\int_0^\infty (t)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(e^{-st})dt$	Laplace integral of $g(t)=t$.
$= -\frac{d}{ds}\int_0^\infty (1)e^{-st}dt$	Use $\int \frac{d}{ds}F(t,s)dt = \frac{d}{ds}\int F(t,s)dt$.
$= -\frac{d}{ds}(1/s)$	Use $\mathcal{L}(1)=1/s$.
$= 1/s^2$	Differentiate.
$\int_0^\infty (t^2)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(te^{-st})dt$	Laplace integral of $g(t)=t^2$.
$= -\frac{d}{ds}\int_0^\infty (t)e^{-st}dt$	
$= -\frac{d}{ds}(1/s^2)$	Use $\mathcal{L}(t)=1/s^2$.
$= 2/s^3$	

Table 1. The Laplace integral $\int_0^\infty g(t)e^{-st}dt$ for $g(t)=1$, t and t^2.

$$\int_0^\infty (1)e^{-st}\,dt = \frac{1}{s} \qquad \int_0^\infty (t)e^{-st}\,dt = \frac{1}{s^2} \qquad \int_0^\infty (t^2)e^{-st}\,dt = \frac{2}{s^3}$$

$$\text{In summary,}\quad \mathcal{L}(t^n) = \frac{n!}{s^{1+n}}$$

An Illustration. The ideas of the **Laplace method** will be illustrated for the solution $y(t)=-t$ of the problem $y'=-1$, $y(0)=0$. The method, entirely different from variation of parameters or undetermined coefficients, uses basic calculus and college algebra; see Table 2.

Table 2. Laplace method details for the illustration $y'=-1$, $y(0)=0$.

$y'(t)e^{-st} = -e^{-st}$	Multiply $y'=-1$ by e^{-st}.
$\int_0^\infty y'(t)e^{-st}dt = \int_0^\infty -e^{-st}dt$	Integrate $t=0$ to $t=\infty$.
$\int_0^\infty y'(t)e^{-st}dt = -1/s$	Use Table 1.
$s\int_0^\infty y(t)e^{-st}dt - y(0) = -1/s$	Integrate by parts on the left.
$\int_0^\infty y(t)e^{-st}dt = -1/s^2$	Use $y(0)=0$ and divide.
$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty (-t)e^{-st}dt$	Use Table 1.
$y(t) = -t$	Apply Lerch's cancellation law.

In Lerch's law, the formal rule of erasing the integral signs is valid *provided* the integrals are equal for large s and certain conditions hold on y and f – see Theorem 2. The illustration in Table 2 shows that Laplace theory requires an in-depth study of a **special integral table**, a table which is a true extension of the usual table found on the inside covers of calculus books. Some entries for the special integral table appear in Table 1 and also in section 7.2, Table 4.

The $\mathcal{L}$-notation for the direct Laplace transform produces briefer details, as witnessed by the translation of Table 2 into Table 3 below. The reader is advised to move from Laplace integral notation to the $\mathcal{L}$–notation as soon as possible, in order to clarify the ideas of the transform method.

Table 3. Laplace method $\mathcal{L}$-notation details for $y' = -1$, $y(0) = 0$ translated from Table 2.

$\mathcal{L}(y'(t)) = \mathcal{L}(-1)$	Apply $\mathcal{L}$ across $y' = -1$, or multiply $y' = -1$ by e^{-st}, integrate $t = 0$ to $t = \infty$.
$\mathcal{L}(y'(t)) = -1/s$	Use Table 1.
$s\mathcal{L}(y(t)) - y(0) = -1/s$	Integrate by parts on the left.
$\mathcal{L}(y(t)) = -1/s^2$	Use $y(0) = 0$ and divide.
$\mathcal{L}(y(t)) = \mathcal{L}(-t)$	Apply Table 1.
$y(t) = -t$	Invoke Lerch's cancellation law.

Some Transform Rules. The formal properties of calculus integrals plus the integration by parts formula used in Tables 2 and 3 leads to these **rules** for the Laplace transform:

$\mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	The integral of a sum is the sum of the integrals.
$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	Constants c pass through the integral sign.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	The t-derivative rule, or integration by parts. See Theorem 3.
$\mathcal{L}(y(t)) = \mathcal{L}(f(t))$ implies $y(t) = f(t)$	Lerch's cancellation law. See Theorem 2.

1 Example (Laplace method) Solve by Laplace's method the initial value problem $y' = 5 - 2t$, $y(0) = 1$.

Solution: Laplace's method is outlined in Tables 2 and 3. The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$\mathcal{L}(y'(t)) = \mathcal{L}(5 - 2t)$	Apply $\mathcal{L}$ across $y' = 5 - 2t$.
$\mathcal{L}(y'(t)) = \dfrac{5}{s} - \dfrac{2}{s^2}$	Use Table 1.
$s\mathcal{L}(y(t)) - y(0) = \dfrac{5}{s} - \dfrac{2}{s^2}$	Apply the t-derivative rule, page 250.
$\mathcal{L}(y(t)) = \dfrac{1}{s} + \dfrac{5}{s^2} - \dfrac{2}{s^3}$	Use $y(0) = 1$ and divide.
$\mathcal{L}(y(t)) = \mathcal{L}(1) + 5\mathcal{L}(t) - \mathcal{L}(t^2)$	Apply Table 1, backwards.
$= \mathcal{L}(1 + 5t - t^2)$	Linearity, page 250.
$y(t) = 1 + 5t - t^2$	Invoke Lerch's cancellation law.

2 Example (Laplace method) Solve by Laplace's method the initial value problem $y'' = 10$, $y(0) = y'(0) = 0$.

Solution: The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 5t^2$.

$\mathcal{L}(y''(t)) = \mathcal{L}(10)$	Apply $\mathcal{L}$ across $y'' = 10$.
$s\mathcal{L}(y'(t)) - y'(0) = \mathcal{L}(10)$	Apply the t-derivative rule to y', that is, replace y by y' on page 250.
$s[s\mathcal{L}(y(t)) - y(0)] - y'(0) = \mathcal{L}(10)$	Repeat the t-derivative rule, on y.
$s^2\mathcal{L}(y(t)) = \mathcal{L}(10)$	Use $y(0) = y'(0) = 0$.
$\mathcal{L}(y(t)) = \dfrac{10}{s^3}$	Use Table 1. Then divide.
$\mathcal{L}(y(t)) = \mathcal{L}(5t^2)$	Apply Table 1, backwards.
$y(t) = 5t^2$	Invoke Lerch's cancellation law.

Existence of the Transform. The Laplace integral $\int_0^\infty e^{-st} f(t)\,dt$ is known to exist in the sense of the improper integral definition[1]

$$\int_0^\infty g(t)dt = \lim_{N\to\infty} \int_0^N g(t)dt$$

provided $f(t)$ belongs to a class of functions known in the literature as functions of **exponential order**. For this class of functions the relation

$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \tag{2}$$

is required to hold for some real number a, or equivalently, for some constants M and α,

$$|f(t)| \le Me^{\alpha t}. \tag{3}$$

In addition, $f(t)$ is required to be **piecewise continuous** on each finite subinterval of $0 \le t < \infty$, a term defined as follows.

[1] An advanced calculus background is assumed for the Laplace transform existence proof. Applications of Laplace theory require only a calculus background.

Definition 1 (piecewise continuous)
A function $f(t)$ is **piecewise continuous** on a finite interval $[a,b]$ provided there exists a partition $a = t_0 < \cdots < t_n = b$ of the interval $[a,b]$ and functions $f_1, f_2, \ldots, f_n$ continuous on $(-\infty,\infty)$ such that for t not a partition point

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1, \\ \vdots & \vdots \\ f_n(t) & t_{n-1} < t < t_n. \end{cases} \tag{4}$$

The values of f at partition points are undecided by equation (4). In particular, equation (4) implies that $f(t)$ has one-sided limits at each point of $a < t < b$ and appropriate one-sided limits at the endpoints. Therefore, f has at worst a **jump discontinuity** at each partition point.

3 **Example (Exponential order)** Show that $f(t) = e^t \cos t + t$ is of exponential order, that is, show that $f(t)$ is piecewise continuous and find $\alpha > 0$ such that $\lim_{t\to\infty} f(t)/e^{\alpha t} = 0$.

Solution: Already, $f(t)$ is continuous, hence piecewise continuous. From L'Hospital's rule in calculus, $\lim_{t\to\infty} p(t)/e^{\alpha t} = 0$ for any polynomial p and any $\alpha > 0$. Choose $\alpha = 2$, then

$$\lim_{t\to\infty} \frac{f(t)}{e^{2t}} = \lim_{t\to\infty} \frac{\cos t}{e^t} + \lim_{t\to\infty} \frac{t}{e^{2t}} = 0.$$

Theorem 1 (Existence of $\mathcal{L}(f)$)
Let $f(t)$ be piecewise continuous on every finite interval in $t \geq 0$ and satisfy $|f(t)| \leq Me^{\alpha t}$ for some constants M and α. Then $\mathcal{L}(f(t))$ exists for $s > \alpha$ and $\lim_{s\to\infty} \mathcal{L}(f(t)) = 0$.

Proof: It has to be shown that the Laplace integral of f is finite for $s > \alpha$. Advanced calculus implies that it is sufficient to show that the integrand is absolutely bounded above by an integrable function $g(t)$. Take $g(t) = Me^{-(s-\alpha)t}$. Then $g(t) \geq 0$. Furthermore, g is integrable, because

$$\int_0^\infty g(t)dt = \frac{M}{s-\alpha}.$$

Inequality $|f(t)| \leq Me^{\alpha t}$ implies the absolute value of the Laplace transform integrand $f(t)e^{-st}$ is estimated by

$$\left|f(t)e^{-st}\right| \leq Me^{\alpha t}e^{-st} = g(t).$$

The limit statement follows from $|\mathcal{L}(f(t))| \leq \int_0^\infty g(t)dt = \dfrac{M}{s-\alpha}$, because the right side of this inequality has limit zero at $s = \infty$. The proof is complete.

Theorem 2 (Lerch)
If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and $\int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f_2(t)e^{-st}dt$ for all $s > s_0$, then $f_1(t) = f_2(t)$ for $t \ge 0$.

Proof: See Widder [**?**].

Theorem 3 (t-Derivative Rule)
If $f(t)$ is continuous, $\lim_{t\to\infty} f(t)e^{-st} = 0$ for all large values of s and $f'(t)$ is piecewise continuous, then $\mathcal{L}(f'(t))$ exists for all large s and $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$.

Proof: See page 278.

Exercises 7.1

Laplace method. Solve the given initial value problem using Laplace's method.

1. $y' = -2$, $y(0) = 0$.

2. $y' = 1$, $y(0) = 0$.

3. $y' = -t$, $y(0) = 0$.

4. $y' = t$, $y(0) = 0$.

5. $y' = 1 - t$, $y(0) = 0$.

6. $y' = 1 + t$, $y(0) = 0$.

7. $y' = 3 - 2t$, $y(0) = 0$.

8. $y' = 3 + 2t$, $y(0) = 0$.

9. $y'' = -2$, $y(0) = y'(0) = 0$.

10. $y'' = 1$, $y(0) = y'(0) = 0$.

11. $y'' = 1 - t$, $y(0) = y'(0) = 0$.

12. $y'' = 1 + t$, $y(0) = y'(0) = 0$.

13. $y'' = 3 - 2t$, $y(0) = y'(0) = 0$.

14. $y'' = 3 + 2t$, $y(0) = y'(0) = 0$.

Exponential order. Show that $f(t)$ is of exponential order, by finding a constant $\alpha \ge 0$ in each case such that $\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$.

15. $f(t) = 1 + t$

16. $f(t) = e^t \sin(t)$

17. $f(t) = \sum_{n=0}^N c_n x^n$, for any choice of the constants $c_0, \ldots, c_N$.

18. $f(t) = \sum_{n=1}^N c_n \sin(nt)$, for any choice of the constants $c_1, \ldots, c_N$.

Existence of transforms. Let $f(t) = te^{t^2}\sin(e^{t^2})$. Establish these results.

19. The function $f(t)$ is not of exponential order.

20. The Laplace integral of $f(t)$, $\int_0^\infty f(t)e^{-st}dt$, converges for all $s > 0$.

Jump Magnitude. For f piecewise continuous, define the **jump** at t by

$$J(t) = \lim_{h\to 0+} f(t+h) - \lim_{h\to 0+} f(t-h).$$

Compute $J(t)$ for the following f.

21. $f(t) = 1$ for $t \ge 0$, else $f(t) = 0$

22. $f(t) = 1$ for $t \ge 1/2$, else $f(t) = 0$

23. $f(t) = t/|t|$ for $t \ne 0$, $f(0) = 0$

24. $f(t) = \sin t/|\sin t|$ for $t \ne n\pi$, $f(n\pi) = (-1)^n$

Taylor series. The series relation $\mathcal{L}(\sum_{n=0}^\infty c_n t^n) = \sum_{n=0}^\infty c_n \mathcal{L}(t^n)$ often holds, in which case the result $\mathcal{L}(t^n) = n! s^{-1-n}$ can be employed to find a series representation of the Laplace transform. Use this idea on the following to find a series formula for $\mathcal{L}(f(t))$.

25. $f(t) = e^{2t} = \sum_{n=0}^\infty (2t)^n/n!$

26. $f(t) = e^{-t} = \sum_{n=0}^\infty (-t)^n/n!$

7.2 Laplace Integral Table

The objective in developing a table of Laplace integrals, e.g., Tables 4 and 5, is to keep the table size small. Table manipulation rules appearing in Table 6, page 259, effectively increase the table size manyfold, making it possible to solve typical differential equations from electrical and mechanical problems. The combination of Laplace tables plus the table manipulation rules is called the **Laplace transform calculus**.

Table 4 is considered to be a table of minimum size to be memorized. Table 5 adds a number of special-use entries. For instance, the Heaviside entry in Table 5 is memorized, but usually not the others.

Derivations are postponed to page 272. The theory of the **gamma function** $\Gamma(x)$ appears below on page 257.

Table 4. A minimal Laplace integral table with $\mathcal{L}$-notation

$\int_0^\infty (t^n)e^{-st}\,dt = \dfrac{n!}{s^{1+n}}$	$\mathcal{L}(t^n) = \dfrac{n!}{s^{1+n}}$
$\int_0^\infty (e^{at})e^{-st}\,dt = \dfrac{1}{s-a}$	$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$
$\int_0^\infty (\cos bt)e^{-st}\,dt = \dfrac{s}{s^2+b^2}$	$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$
$\int_0^\infty (\sin bt)e^{-st}\,dt = \dfrac{b}{s^2+b^2}$	$\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

Table 5. Laplace integral table extension

$\mathcal{L}(H(t-a)) = \dfrac{e^{-as}}{s}\ (a \ge 0)$	Heaviside unit step, defined by $H(t) = \begin{cases} 1 & \text{for } t \ge 0, \\ 0 & \text{otherwise.} \end{cases}$
$\mathcal{L}(\delta(t-a)) = e^{-as}$	Dirac delta, $\delta(t) = dH(t)$. Special usage rules apply.
$\mathcal{L}(\textbf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$	Staircase function, $\textbf{floor}(x)$ = greatest integer $\le x$.
$\mathcal{L}(\textbf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$	Square wave, $\textbf{sqw}(x) = (-1)^{\textbf{floor}(x)}$.
$\mathcal{L}(a\,\textbf{trw}(t/a)) = \dfrac{1}{s^2}\tanh(as/2)$	Triangular wave, $\textbf{trw}(x) = \int_0^x \textbf{sqw}(r)dr$.
$\mathcal{L}(t^\alpha) = \dfrac{\Gamma(1+\alpha)}{s^{1+\alpha}}$	Generalized power function, $\Gamma(1+\alpha) = \int_0^\infty e^{-x}x^\alpha dx$.
$\mathcal{L}(t^{-1/2}) = \sqrt{\dfrac{\pi}{s}}$	Because $\Gamma(1/2) = \sqrt{\pi}$.

4 Example (Laplace transform) Let $f(t) = t(t-1) - \sin 2t + e^{3t}$. Compute $\mathcal{L}(f(t))$ using the basic Laplace table and transform linearity properties.

Solution:

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(t^2 - 5t - \sin 2t + e^{3t}) && \text{Expand } t(t-5). \\ &= \mathcal{L}(t^2) - 5\mathcal{L}(t) - \mathcal{L}(\sin 2t) + \mathcal{L}(e^{3t}) && \text{Linearity applied.} \\ &= \frac{2}{s^3} - \frac{5}{s^2} - \frac{2}{s^2+4} + \frac{1}{s-3} && \text{Table lookup.} \end{aligned}$$

5 Example (Inverse Laplace transform) Use the basic Laplace table backwards plus transform linearity properties to solve for $f(t)$ in the equation

$$\mathcal{L}(f(t)) = \frac{s}{s^2+16} + \frac{2}{s-3} + \frac{s+1}{s^3}.$$

Solution:

$$\begin{aligned} \mathcal{L}(f(t)) &= \frac{s}{s^2+16} + 2\frac{1}{s-3} + \frac{1}{s^2} + \frac{1}{2}\frac{2}{s^3} && \text{Convert to table entries.} \\ &= \mathcal{L}(\cos 4t) + 2\mathcal{L}(e^{3t}) + \mathcal{L}(t) + \tfrac{1}{2}\mathcal{L}(t^2) && \text{Laplace table (backwards).} \\ &= \mathcal{L}(\cos 4t + 2e^{3t} + t + \tfrac{1}{2}t^2) && \text{Linearity applied.} \\ f(t) &= \cos 4t + 2e^{3t} + t + \tfrac{1}{2}t^2 && \text{Lerch's cancellation law.} \end{aligned}$$

6 Example (Heaviside) Find the Laplace transform of $f(t)$ in Figure 1.

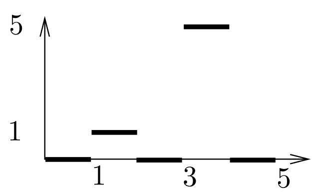

Figure 1. A piecewise defined function $f(t)$ on $0 \le t < \infty$: $f(t) = 0$ except for $1 \le t < 2$ and $3 \le t < 4$.

Solution: The details require the use of the Heaviside function formula

$$H(t-a) - H(t-b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$$

The formula for $f(t)$:

$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 5 & 3 \le t < 4, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 1 \le t < 2, \\ 0 & \text{otherwise} \end{cases} + 5 \begin{cases} 1 & 3 \le t < 4, \\ 0 & \text{otherwise} \end{cases}$$

Then $f(t) = f_1(t) + 5f_2(t)$ where $f_1(t) = H(t-1) - H(t-2)$ and $f_2(t) = H(t-3) - H(t-4)$. The extended table gives

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(f_1(t)) + 5\mathcal{L}(f_2(t)) && \text{Linearity.} \\ &= \mathcal{L}(H(t-1)) - \mathcal{L}(H(t-2)) + 5\mathcal{L}(f_2(t)) && \text{Substitute for } f_1. \end{aligned}$$

$$= \frac{e^{-s} - e^{-2s}}{s} + 5\mathcal{L}(f_2(t)) \qquad \text{Extended table used.}$$
$$= \frac{e^{-s} - e^{-2s} + 5e^{-3s} - 5e^{-4s}}{s} \qquad \text{Similarly for } f_2.$$

7 Example (Dirac delta) A machine shop tool that repeatedly hammers a die is modeled by the Dirac impulse model $f(t) = \sum_{n=1}^{N} \delta(t-n)$. Show that $\mathcal{L}(f(t)) = \sum_{n=1}^{N} e^{-ns}$.

Solution:

$$\mathcal{L}(f(t)) = \mathcal{L}\left(\sum_{n=1}^{N} \delta(t-n)\right)$$
$$= \sum_{n=1}^{N} \mathcal{L}(\delta(t-n)) \qquad \text{Linearity.}$$
$$= \sum_{n=1}^{N} e^{-ns} \qquad \text{Extended Laplace table.}$$

8 Example (Square wave) A periodic camshaft force $f(t)$ applied to a mechanical system has the idealized graph shown in Figure 2. Show that $f(t) = 1 + \textbf{sqw}(t)$ and $\mathcal{L}(f(t)) = \frac{1}{s}(1 + \tanh(s/2))$.

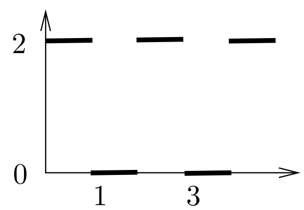

Figure 2. A periodic force $f(t)$ applied to a mechanical system.

Solution:

$$1 + \textbf{sqw}(t) = \begin{cases} 1+1 & 2n \le t < 2n+1, \ n = 0, 1, \ldots, \\ 1-1 & 2n+1 \le t < 2n+2, \ n = 0, 1, \ldots, \end{cases}$$
$$= \begin{cases} 2 & 2n \le t < 2n+1, \ n = 0, 1, \ldots, \\ 0 & \text{otherwise}, \end{cases}$$
$$= f(t).$$

By the extended Laplace table, $\mathcal{L}(f(t)) = \mathcal{L}(1) + \mathcal{L}(\textbf{sqw}(t)) = \frac{1}{s} + \frac{\tanh(s/2)}{s}$.

9 Example (Sawtooth wave) Express the P-periodic sawtooth wave represented in Figure 3 as $f(t) = ct/P - c\,\textbf{floor}(t/P)$ and obtain the formula

$$\mathcal{L}(f(t)) = \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}}.$$

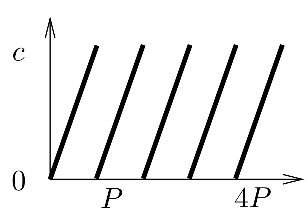

Figure 3. A P-periodic sawtooth wave $f(t)$ of height $c > 0$.

Solution: The representation originates from geometry, because the periodic function f can be viewed as derived from ct/P by subtracting the correct constant from each of intervals $[P, 2P]$, $[2P, 3P]$, etc.

The technique used to verify the identity is to define $g(t) = ct/P - c\,\textbf{floor}(t/P)$ and then show that g is P-periodic and $f(t) = g(t)$ on $0 \le t < P$. Two P-periodic functions equal on the base interval $0 \le t < P$ have to be identical, hence the representation follows.

The fine details: for $0 \le t < P$, $\textbf{floor}(t/P) = 0$ and $\textbf{floor}(t/P + k) = k$. Hence $g(t + kP) = ct/P + ck - c\,\textbf{floor}(k) = ct/P = g(t)$, which implies that g is P-periodic and $g(t) = f(t)$ for $0 \le t < P$.

$$\begin{aligned}\mathcal{L}(f(t)) &= \frac{c}{P}\mathcal{L}(t) - c\mathcal{L}(\textbf{floor}(t/P)) && \text{Linearity.}\\ &= \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}} && \text{Basic and extended table applied.}\end{aligned}$$

10 Example (Triangular wave) Express the triangular wave f of Figure 4 in terms of the square wave **sqw** and obtain $\mathcal{L}(f(t)) = \dfrac{5}{\pi s^2}\tanh(\pi s/2)$.

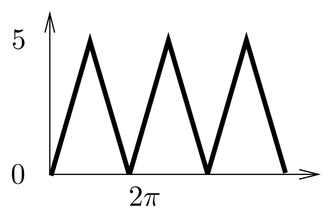

Figure 4. A 2π-periodic triangular wave $f(t)$ of height 5.

Solution: The representation of f in terms of **sqw** is $f(t) = 5\int_0^{t/\pi} \textbf{sqw}(x)dx$.

Details: A 2-periodic triangular wave of height 1 is obtained by integrating the square wave of period 2. A wave of height c and period 2 is given by $c\,\textbf{trw}(t) = c\int_0^t \textbf{sqw}(x)dx$. Then $f(t) = c\,\textbf{trw}(2t/P) = c\int_0^{2t/P} \textbf{sqw}(x)dx$ where $c = 5$ and $P = 2\pi$.

Laplace transform details: Use the extended Laplace table as follows.

$$\mathcal{L}(f(t)) = \frac{5}{\pi}\mathcal{L}(\pi\,\textbf{trw}(t/\pi)) = \frac{5}{\pi s^2}\tanh(\pi s/2).$$

Gamma Function.

In mathematical physics, the **Gamma function** or the **generalized factorial function** is given by the identity

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt, \quad x > 0. \tag{1}$$

This function is tabulated and available in computer languages like Fortran, C and C++. It is also available in computer algebra systems and numerical laboratories. Some useful properties of $\Gamma(x)$:

$$\Gamma(1+x) = x\Gamma(x) \tag{2}$$

$$\Gamma(1+n) = n! \quad \text{for integers} \quad n \ge 1. \tag{3}$$

Details for relations (2) and (3): Start with $\int_0^\infty e^{-t}dt = 1$, which gives $\Gamma(1) = 1$. Use this identity and successively relation (2) to obtain relation (3). To prove identity (2), integration by parts is applied, as follows:

$\Gamma(1+x) = \int_0^\infty e^{-t}t^x dt$	Definition.
$= -t^x e^{-t}\vert_{t=0}^{t=\infty} + \int_0^\infty e^{-t}xt^{x-1}dt$	Use $u = t^x$, $dv = e^{-t}dt$.
$= x\int_0^\infty e^{-t}t^{x-1}dt$	Boundary terms are zero for $x > 0$.
$= x\Gamma(x)$.	

Exercises 7.2

Laplace transform. Compute $\mathcal{L}(f(t))$ using the basic Laplace table and the linearity properties of the transform. Do not use the direct Laplace transform!

1. $\mathcal{L}(2t)$

2. $\mathcal{L}(4t)$

3. $\mathcal{L}(1+2t+t^2)$

4. $\mathcal{L}(t^2-3t+10)$

5. $\mathcal{L}(\sin 2t)$

6. $\mathcal{L}(\cos 2t)$

7. $\mathcal{L}(e^{2t})$

8. $\mathcal{L}(e^{-2t})$

9. $\mathcal{L}(t+\sin 2t)$

10. $\mathcal{L}(t-\cos 2t)$

11. $\mathcal{L}(t+e^{2t})$

12. $\mathcal{L}(t-3e^{-2t})$

13. $\mathcal{L}((t+1)^2)$

14. $\mathcal{L}((t+2)^2)$

15. $\mathcal{L}(t(t+1))$

16. $\mathcal{L}((t+1)(t+2))$

17. $\mathcal{L}(\sum_{n=0}^{10} t^n/n!)$

18. $\mathcal{L}(\sum_{n=0}^{10} t^{n+1}/n!)$

19. $\mathcal{L}(\sum_{n=1}^{10} \sin nt)$

20. $\mathcal{L}(\sum_{n=0}^{10} \cos nt)$

Inverse Laplace transform. Solve the given equation for the function $f(t)$. Use the basic table and linearity properties of the Laplace transform.

21. $\mathcal{L}(f(t)) = s^{-2}$

22. $\mathcal{L}(f(t)) = 4s^{-2}$

23. $\mathcal{L}(f(t)) = 1/s + 2/s^2 + 3/s^3$

24. $\mathcal{L}(f(t)) = 1/s^3 + 1/s$

25. $\mathcal{L}(f(t)) = 2/(s^2+4)$

26. $\mathcal{L}(f(t)) = s/(s^2+4)$

27. $\mathcal{L}(f(t)) = 1/(s-3)$

28. $\mathcal{L}(f(t)) = 1/(s+3)$

29. $\mathcal{L}(f(t)) = 1/s + s/(s^2+4)$

30. $\mathcal{L}(f(t)) = 2/s - 2/(s^2+4)$

31. $\mathcal{L}(f(t)) = 1/s + 1/(s-3)$

32. $\mathcal{L}(f(t)) = 1/s - 3/(s-2)$

33. $\mathcal{L}(f(t)) = (2+s)^2/s^3$

34. $\mathcal{L}(f(t)) = (s+1)/s^2$

35. $\mathcal{L}(f(t)) = s(1/s^2 + 2/s^3)$

36. $\mathcal{L}(f(t)) = (s+1)(s-1)/s^3$

37. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n}$

38. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{2+n}$

39. $\mathcal{L}(f(t)) = \sum_{n=1}^{10} \dfrac{n}{s^2+n^2}$

40. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \dfrac{s}{s^2+n^2}$

7.3 Laplace Transform Rules

In Table 6, the basic table manipulation rules are summarized. Full statements and proofs of the rules appear in section 7.7, page 277.

The rules are applied here to several key examples. Partial fraction expansions do not appear here, but in section 7.4, in connection with Heaviside's coverup method.

Table 6. Laplace transform rules

$\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	Linearity. The Laplace of a sum is the sum of the Laplaces.
$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	Linearity. Constants move through the $\mathcal{L}$-symbol.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	The t-derivative rule. Derivatives $\mathcal{L}(y')$ are replaced in transformed equations.
$\mathcal{L}\left(\int_0^t g(x)dx\right) = \frac{1}{s}\mathcal{L}(g(t))$	The t-integral rule.
$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$	The s-differentiation rule. Multiplying f by t applies $-d/ds$ to the transform of f.
$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\vert_{s\to(s-a)}$	First shifting rule. Multiplying f by e^{at} replaces s by $s-a$.
$\mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t))$, $\mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a))$	Second shifting rule. First and second forms.
$\mathcal{L}(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1-e^{-Ps}}$	Rule for P-periodic functions. Assumed here is $f(t+P) = f(t)$.
$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}((f*g)(t))$	Convolution rule. Define $(f*g)(t) = \int_0^t f(x)g(t-x)dx$.

11 **Example (Harmonic oscillator)** Solve by Laplace's method the initial value problem $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$.

Solution: The solution is $x(t) = \sin t$. The details:

$\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(0)$	Apply $\mathcal{L}$ across the equation.
$s\mathcal{L}(x') - x'(0) + \mathcal{L}(x) = 0$	Use the t-derivative rule.
$s[s\mathcal{L}(x) - x(0)] - x'(0) + \mathcal{L}(x) = 0$	Use again the t-derivative rule.
$(s^2+1)\mathcal{L}(x) = 1$	Use $x(0) = 0$, $x'(0) = 1$.
$\mathcal{L}(x) = \frac{1}{s^2+1}$	Divide.
$= \mathcal{L}(\sin t)$	Basic Laplace table.
$x(t) = \sin t$	Invoke Lerch's cancellation law.

12 Example (s-differentiation rule) Show the steps for $\mathcal{L}(t^2\, e^{5t}) = \dfrac{2}{(s-5)^3}$.

Solution:

$\mathcal{L}(t^2 e^{5t}) = \left(-\dfrac{d}{ds}\right)\left(-\dfrac{d}{ds}\right)\mathcal{L}(e^{5t})$	Apply s-differentiation.
$= (-1)^2 \dfrac{d}{ds}\dfrac{d}{ds}\left(\dfrac{1}{s-5}\right)$	Basic Laplace table.
$= \dfrac{d}{ds}\left(\dfrac{-1}{(s-5)^2}\right)$	Calculus power rule.
$= \dfrac{2}{(s-5)^3}$	Identity verified.

13 Example (First shifting rule) Show the steps for $\mathcal{L}(t^2\, e^{-3t}) = \dfrac{2}{(s+3)^3}$.

Solution:

$\mathcal{L}(t^2 e^{-3t}) = \mathcal{L}(t^2)\big\|_{s\to s-(-3)}$	First shifting rule.
$= \left(\dfrac{2}{s^{2+1}}\right)\Big\|_{s\to s-(-3)}$	Basic Laplace table.
$= \dfrac{2}{(s+3)^3}$	Identity verified.

14 Example (Second shifting rule) Show the steps for

$$\mathcal{L}(\sin t\, H(t-\pi)) = \frac{e^{-\pi s}}{s^2+1}.$$

Solution: The second shifting rule is applied as follows.

$\mathcal{L}(\sin t\, H(t-\pi)) = \mathcal{L}(g(t)H(t-a)$	Choose $g(t) = \sin t$, $a = \pi$.
$= e^{-as}\mathcal{L}(g(t+a)$	Second form, second shifting theorem.
$= e^{-\pi s}\mathcal{L}(\sin(t+\pi))$	Substitute $a = \pi$.
$= e^{-\pi s}\mathcal{L}(-\sin t)$	Sum rule $\sin(a+b) = \sin a \cos b + \sin b \cos a$ plus $\sin \pi = 0$, $\cos \pi = -1$.
$= e^{-\pi s}\dfrac{-1}{s^2+1}$	Basic Laplace table. Identity verified.

15 Example (Trigonometric formulas) Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(t \cos at) = \dfrac{s^2 - a^2}{(s^2+a^2)^2}$

(b) $\mathcal{L}(t \sin at) = \dfrac{2sa}{(s^2+a^2)^2}$

(c) $\mathcal{L}(t^2 \cos at) = \dfrac{2(s^3 - 3sa^2)}{(s^2+a^2)^3}$

(d) $\mathcal{L}(t^2 \sin at) = \dfrac{6s^2 a - a^3}{(s^2+a^2)^3}$

Solution: The details for **(a)**:

$$\begin{aligned}
\mathcal{L}(t\cos at) &= -(d/ds)\mathcal{L}(\cos at) && \text{Use } s\text{-differentiation.}\\
&= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) && \text{Basic Laplace table.}\\
&= \frac{s^2-a^2}{(s^2+a^2)^2} && \text{Calculus quotient rule.}
\end{aligned}$$

The details for **(c)**:

$$\begin{aligned}
\mathcal{L}(t^2\cos at) &= -(d/ds)\mathcal{L}((-t)\cos at) && \text{Use } s\text{-differentiation.}\\
&= \frac{d}{ds}\left(-\frac{s^2-a^2}{(s^2+a^2)^2}\right) && \text{Result of } \textbf{(a)}.\\
&= \frac{2s^3-6sa^2)}{(s^2+a^2)^3} && \text{Calculus quotient rule.}
\end{aligned}$$

The similar details for **(b)** and **(d)** are left as exercises.

16 Example (Exponentials) Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$ **(c)** $\mathcal{L}(te^{at}\cos bt) = \dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$

(b) $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ **(d)** $\mathcal{L}(te^{at}\sin bt) = \dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

Solution: Details for **(a)**:

$$\begin{aligned}
\mathcal{L}(e^{at}\cos bt) &= \mathcal{L}(\cos bt)|_{s\to s-a} && \text{First shifting rule.}\\
&= \left(\frac{s}{s^2+b^2}\right)\Bigg|_{s\to s-a} && \text{Basic Laplace table.}\\
&= \frac{s-a}{(s-a)^2+b^2} && \text{Verified } \textbf{(a)}.
\end{aligned}$$

Details for **(c)**:

$$\begin{aligned}
\mathcal{L}(te^{at}\cos bt) &= \mathcal{L}(t\cos bt)|_{s\to s-a} && \text{First shifting rule.}\\
&= \left(-\frac{d}{ds}\mathcal{L}(\cos bt)\right)\Bigg|_{s\to s-a} && \text{Apply } s\text{-differentiation.}\\
&= \left(-\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)\right)\Bigg|_{s\to s-a} && \text{Basic Laplace table.}\\
&= \left(\frac{s^2-b^2}{(s^2+b^2)^2}\right)\Bigg|_{s\to s-a} && \text{Calculus quotient rule.}\\
&= \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2} && \text{Verified } \textbf{(c)}.
\end{aligned}$$

Left as exercises are **(b)** and **(d)**.

17 **Example (Hyperbolic functions)** Establish these Laplace transform facts about $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$.

(a) $\mathcal{L}(\cosh at) = \dfrac{s}{s^2 - a^2}$

(b) $\mathcal{L}(\sinh at) = \dfrac{a}{s^2 - a^2}$

(c) $\mathcal{L}(t \cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

(d) $\mathcal{L}(t \sinh at) = \dfrac{2as}{(s^2 - a^2)^2}$

Solution: The details for **(a)**:

$$\begin{aligned}
\mathcal{L}(\cosh at) &= \tfrac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) && \text{Definition plus linearity of } \mathcal{L}. \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) && \text{Basic Laplace table.} \\
&= \frac{s}{s^2 - a^2} && \text{Identity (a) verified.}
\end{aligned}$$

The details for **(d)**:

$$\begin{aligned}
\mathcal{L}(t \sinh at) &= -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right) && \text{Apply the } s\text{-differentiation rule.} \\
&= \frac{a(2s)}{(s^2 - a^2)^2} && \text{Calculus power rule; (d) verified.}
\end{aligned}$$

Left as exercises are **(b)** and **(c)**.

18 **Example (s-differentiation)** Solve $\mathcal{L}(f(t)) = \dfrac{2s}{(s^2+1)^2}$ for $f(t)$.

Solution: The solution is $f(t) = t \sin t$. The details:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{2s}{(s^2+1)^2} \\
&= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) && \text{Calculus power rule } (u^n)' = nu^{n-1}u'. \\
&= -\frac{d}{ds}\left(\mathcal{L}(\sin t)\right) && \text{Basic Laplace table.} \\
&= \mathcal{L}(t \sin t) && \text{Apply the } s\text{-differentiation rule.} \\
f(t) &= t \sin t && \text{Lerch's cancellation law.}
\end{aligned}$$

19 **Example (First shift rule)** Solve $\mathcal{L}(f(t)) = \dfrac{s+2}{2^2 + 2s + 2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-t}\cos t + e^{-t}\sin t$. The details:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s+2}{s^2 + 2s + 2} && \text{Signal for this method: the denominator has complex roots.} \\
&= \frac{s+2}{(s+1)^2 + 1} && \text{Complete the square, denominator.}
\end{aligned}$$

$= \dfrac{S+1}{S^2+1}$	Substitute S for $s+1$.
$= \dfrac{S}{S^2+1} + \dfrac{1}{S^2+1}$	Split into Laplace table entries.
$= \mathcal{L}(\cos t) + \mathcal{L}(\sin t)\big\|_{s\to S=s+1}$	Basic Laplace table.
$= \mathcal{L}(e^{-t}\cos t) + \mathcal{L}(e^{-t}\sin t)$	First shift rule.
$f(t) = e^{-t}\cos t + e^{-t}\sin t$	Invoke Lerch's cancellation law.

20 Example (Damped oscillator) Solve by Laplace's method the initial value problem $x'' + 2x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$.

Solution: The solution is $x(t) = e^{-t}\cos t$. The details:

$\mathcal{L}(x'') + 2\mathcal{L}(x') + 2\mathcal{L}(x) = \mathcal{L}(0)$	Apply $\mathcal{L}$ across the equation.
$s\mathcal{L}(x') - x'(0) + 2\mathcal{L}(x') + 2\mathcal{L}(x) = 0$	The t-derivative rule on x'.
$s[s\mathcal{L}(x) - x(0)] - x'(0)$ $+2[\mathcal{L}(x) - x(0)] + 2\mathcal{L}(x) = 0$	The t-derivative rule on x.
$(s^2+2s+2)\mathcal{L}(x) = 1 + s$	Use $x(0) = 1$, $x'(0) = -1$.
$\mathcal{L}(x) = \dfrac{s+1}{s^2+2s+2}$	Divide.
$= \dfrac{s+1}{(s+1)^2+1}$	Complete the square in the denominator.
$= \mathcal{L}(\cos t)\big\|_{s\to s+1}$	Basic Laplace table.
$= \mathcal{L}(e^{-t}\cos t)$	First shifting rule.
$x(t) = e^{-t}\cos t$	Invoke Lerch's cancellation law.

21 Example (Rectified sine wave) Compute the Laplace transform of the rectified sine wave $f(t) = |\sin\omega t|$ and show it can be expressed in the form

$$\mathcal{L}(|\sin\omega t|) = \frac{\omega\ \coth\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2}.$$

Solution: The periodic function formula will be applied with period $P = 2\pi/\omega$. The calculation reduces to the evaluation of $J = \int_0^P f(t)e^{-st}dt$. Because $\sin\omega t \le 0$ on $\pi/\omega \le t \le 2\pi/\omega$, integral J can be written as $J = J_1 + J_2$, where

$$J_1 = \int_0^{\pi/\omega} \sin\omega t\, e^{-st}dt, \quad J_2 = \int_{\pi/\omega}^{2\pi/\omega} -\sin\omega t\, e^{-st}dt.$$

Integral tables give the result

$$\int \sin\omega t\, e^{-st}\, dt = -\frac{\omega e^{-st}\cos(\omega t)}{s^2+\omega^2} - \frac{s e^{-st}\sin(\omega t)}{s^2+\omega^2}.$$

Then

$$J_1 = \frac{\omega(e^{-\pi * s/\omega}+1)}{s^2+\omega^2}, \quad J_2 = \frac{\omega(e^{-2\pi s/\omega}+e^{-\pi s/\omega})}{s^2+\omega^2},$$

$$J = \frac{\omega(e^{-\pi s/\omega}+1)^2}{s^2+\omega^2}.$$

The remaining challenge is to write the answer for $\mathcal{L}(f(t))$ in terms of coth. The details:

$\mathcal{L}(f(t)) = \dfrac{J}{1-e^{-Ps}}$	Periodic function formula.
$= \dfrac{J}{(1-e^{-Ps/2})(1+e^{-Ps/2})}$	Apply $1-x^2=(1-x)(1+x)$, $x=e^{-Ps/2}$.
$= \dfrac{\omega(1+e^{-Ps/2})}{(1-e^{-Ps/2})(s^2+\omega^2)}$	Cancel factor $1+e^{-Ps/2}$.
$= \dfrac{e^{Ps/4}+e^{-Ps/4}}{e^{Ps/4}-e^{-Ps/4}}\dfrac{\omega}{s^2+\omega^2}$	Factor out $e^{-Ps/4}$, then cancel.
$= \dfrac{2\cosh(Ps/4)}{2\sinh(Ps/4)}\dfrac{\omega}{s^2+\omega^2}$	Apply cosh, sinh identities.
$= \dfrac{\omega\coth(Ps/4)}{s^2+\omega^2}$	Use $\coth u = \cosh u/\sinh u$.
$= \dfrac{\omega\coth\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2}$	Identity verified.

22 Example (Half–wave rectification) Compute the Laplace transform of the half–wave rectification of $\sin\omega t$, denoted $g(t)$, in which the negative cycles of $\sin\omega t$ have been canceled to create $g(t)$. Show in particular that

$$\mathcal{L}(g(t)) = \frac{1}{2}\frac{\omega}{s^2+\omega^2}\left(1+\coth\left(\frac{\pi s}{2\omega}\right)\right)$$

Solution: The half–wave rectification of $\sin\omega t$ is $g(t) = (\sin\omega t + |\sin\omega t|)/2$. Therefore, the basic Laplace table plus the result of Example 21 give

$$\begin{aligned}\mathcal{L}(2g(t)) &= \mathcal{L}(\sin\omega t) + \mathcal{L}(|\sin\omega t|)\\ &= \frac{\omega}{s^2+\omega^2} + \frac{\omega\cosh(\pi s/(2\omega))}{s^2+\omega^2}\\ &= \frac{\omega}{s^2+\omega^2}(1+\cosh(\pi s/(2\omega))\end{aligned}$$

Dividing by 2 produces the identity.

23 Example (Shifting rules) Solve $\mathcal{L}(f(t)) = e^{-3s}\dfrac{s+1}{s^2+2s+2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{3-t}\cos(t-3)H(t-3)$. The details:

$\mathcal{L}(f(t)) = e^{-3s}\dfrac{s+1}{(s+1)^2+1}$	Complete the square.
$= e^{-3s}\dfrac{S}{S^2+1}$	Replace $s+1$ by S.
$= e^{-3S+3}\left.(\mathcal{L}(\cos t))\right\vert_{s\to S=s+1}$	Basic Laplace table.

$$
\begin{aligned}
&= e^3 \left(e^{-3s}\mathcal{L}(\cos t)\right)\Big|_{s\to S=s+1} && \text{Regroup factor } e^{-3S}.\\
&= e^3 \left(\mathcal{L}(\cos(t-3)H(t-3))\right)\big|_{s\to S=s+1} && \text{Second shifting rule.}\\
&= e^3\mathcal{L}(e^{-t}\cos(t-3)H(t-3)) && \text{First shifting rule.}\\
f(t) &= e^{3-t}\cos(t-3)H(t-3) && \text{Lerch's cancellation law.}
\end{aligned}
$$

24 Example () Solve $\mathcal{L}(f(t) = \dfrac{s+7}{s^2+4s+8}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t)$. The details:

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s+7}{(s+2)^2+4} && \text{Complete the square.}\\
&= \frac{S+5}{S^2+4} && \text{Replace } s+2 \text{ by } S.\\
&= \frac{S}{S^2+4} + \frac{5}{2}\frac{2}{S^2+4} && \text{Split into table entries.}\\
&= \left.\frac{s}{s^2+4} + \frac{5}{2}\frac{2}{s^2+4}\right|_{s\to S=s+2} && \text{Prepare for shifting rule.}\\
&= \mathcal{L}(\cos 2t) + \tfrac{5}{2}\mathcal{L}(\sin 2t)\big|_{s\to S=s+2} && \text{Basic Laplace table.}\\
&= \mathcal{L}(e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t)) && \text{First shifting rule.}\\
f(t) &= e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t) && \text{Lerch's cancellation law.}
\end{aligned}
$$

7.4 Heaviside's Method

This practical method was popularized by the English electrical engineer Oliver Heaviside (1850–1925). A typical application of the method is to solve

$$\frac{2s}{(s+1)(s^2+1)} = \mathcal{L}(f(t))$$

for the t-expression $f(t) = -e^{-t} + \cos t + \sin t$. The details in Heaviside's method involve a sequence of easy-to-learn college algebra steps.

More precisely, **Heaviside's method** systematically converts a polynomial quotient

$$(1) \qquad \frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m}$$

into the form $\mathcal{L}(f(t))$ for some expression $f(t)$. It is assumed that $a_0, .., a_n, b_0, \ldots, b_m$ are constants and the polynomial quotient (1) has limit zero at $s = \infty$.

Partial Fraction Theory

In college algebra, it is shown that a rational function (1) can be expressed as the sum of terms of the form

$$(2) \qquad \frac{A}{(s-s_0)^k}$$

where A is a real or complex constant and $(s-s_0)^k$ divides the denominator in (1). In particular, s_0 is a *root* of the denominator in (1).

Assume fraction (1) has **real coefficients**. If s_0 in (2) is real, then A is *real*. If $s_0 = \alpha + i\beta$ in (2) is *complex*, then $(s - \overline{s_0})^k$ also appears, where $\overline{s_0} = \alpha - i\beta$ is the complex conjugate of s_0. The corresponding terms in (2) turn out to be complex conjugates of one another, which can be combined in terms of *real* numbers B and C as

$$(3) \qquad \frac{A}{(s-s_0)^k} + \frac{\overline{A}}{(s-\overline{s_0})^k} = \frac{B + C\,s}{((s-\alpha)^2 + \beta^2)^k}\,.$$

Simple Roots. Assume that (1) has *real coefficients* and the denominator of the fraction (1) has **distinct real roots** $s_1, \ldots, s_N$ and **distinct complex roots** $\alpha_1 + i\beta_1, \ldots, \alpha_M + i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

$$(4) \qquad \frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s - s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s-\alpha_q)}{(s-\alpha_q)^2 + \beta_q^2}\,.$$

Multiple Roots. Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly **multiple roots**. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$, $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

$$(5) \qquad \sum_{p=1}^{N} \sum_{1\le k\le N_p} \frac{A_{p,k}}{(s-s_p)^k} + \sum_{q=1}^{M} \sum_{1\le k\le M_q} \frac{B_{q,k} + C_{q,k}(s-\alpha_q)}{((s-\alpha_q)^2+\beta_q^2)^k}.$$

Heaviside's Coverup Method

The method applies only to the case of distinct roots of the denominator in (1). Extensions to multiple-root cases can be made; see page 268.

To illustrate Oliver Heaviside's ideas, consider the problem details

$$(6) \qquad \begin{aligned} \frac{2s+1}{s(s-1)(s+1)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \\ &= \mathcal{L}(A) + \mathcal{L}(Be^t) + \mathcal{L}(Ce^{-t}) \\ &= \mathcal{L}(A + Be^t + Ce^{-t}) \end{aligned}$$

The first line (6) uses college algebra partial fractions. The second and third lines use the Laplace integral table and properties of $\mathcal{L}$.

Heaviside's mysterious method. Oliver Heaviside proposed to find in (6) the constant $C = \frac{1}{2}$ by a **cover–up method**:

$$\left.\frac{2s+1}{s(s-1)\boxed{}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{}}.$$

The *instructions* are to cover–up the matching factors $(s+1)$ on the left and right with box $\boxed{}$, then evaluate on the left at the *root* s which makes the contents of the box zero. The other terms on the right are replaced by zero.

To justify Heaviside's cover–up method, multiply (6) by the denominator $s+1$ of partial fraction $C/(s+1)$:

$$\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}} = \frac{A\boxed{(s+1)}}{s} + \frac{B\boxed{(s+1)}}{s-1} + \frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.$$

Set $\boxed{(s+1)} = 0$ in the display. Cancellations left and right plus annihilation of two terms on the right gives Heaviside's prescription

$$\left.\frac{2s+1}{s(s-1)}\right|_{s+1=0} = C.$$

The factor $(s+1)$ in (6) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed.

Extension to Multiple Roots. An extension of Heaviside's method is possible for the case of repeated roots. The basic idea is to *factor–out the repeats*. To illustrate, consider the partial fraction expansion details

$R = \dfrac{1}{(s+1)^2(s+2)}$	A sample rational function having repeated roots.
$= \dfrac{1}{s+1}\left(\dfrac{1}{(s+1)(s+2)}\right)$	Factor–out the repeats.
$= \dfrac{1}{s+1}\left(\dfrac{1}{s+1}+\dfrac{-1}{s+2}\right)$	Apply the cover–up method to the simple root fraction.
$= \dfrac{1}{(s+1)^2}+\dfrac{-1}{(s+1)(s+2)}$	Multiply.
$= \dfrac{1}{(s+1)^2}+\dfrac{-1}{s+1}+\dfrac{1}{s+2}$	Apply the cover–up method to the last fraction on the right.

Terms with only one root in the denominator are already partial fractions. Thus the work centers on expansion of quotients in which the denominator has two or more roots.

Special Methods. Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$R = \dfrac{1}{(s+1)^2(s+2)}$	A fraction with multiple roots.
$= \dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s+2}$	See equation (5).
$= \dfrac{A}{s+1}+\dfrac{1}{(s+1)^2}+\dfrac{1}{s+2}$	Find B and C by Heaviside's cover–up method.
$= \dfrac{-1}{s+1}+\dfrac{1}{(s+1)^2}+\dfrac{1}{s+2}$	Multiply by $s+1$. Set $s=\infty$. Then $0 = A+1$.

The illustration works for one root of multiplicity two, because $s=\infty$ will resolve the coefficient not found by the cover–up method.

In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{A_1}{s-s_0}+\frac{A_2}{(s-s_0)^2}+\cdots+\frac{A_k}{(s-s_0)^k}.$$

Heaviside's cover–up method directly finds A_k, but not A_1 to A_{k-1}.

7.5 Heaviside Step and Dirac Delta

Heaviside Function. The **unit step function** or **Heaviside function** is defined by

$$H(x) = \begin{cases} 1 & \text{for} \quad x \ge 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$

The most often–used formula involving the Heaviside function is the **characteristic function** of the interval $a \le t < b$, given by

$$(1) \qquad H(t-a) - H(t-b) = \begin{cases} 1 & a \le t < b, \\ 0 & t < a, \quad t \ge b. \end{cases}$$

To illustrate, a square wave $\mathbf{sqw}(t) = (-1)^{\mathbf{floor}(t)}$ can be written in the series form

$$\sum_{n=0}^{\infty} (-1)^n (H(t-n) - H(t-n-1)).$$

Dirac Delta. A precise mathematical definition of the Dirac delta, denoted δ, is not possible to give here. Following its inventor P. Dirac, the definition should be

$$\delta(t) = dH(t).$$

The latter is nonsensical, because the unit step does not have a calculus derivative at $t = 0$. However, $dH(t)$ could have the meaning of a Riemann-Stieltjes integrator, which restrains $dH(t)$ to have meaning only under an integral sign. It is in this sense that the Dirac delta δ is defined.

What do we mean by the differential equation

$$x'' + 16x = 5\delta(t - t_0)?$$

The equation $x'' + 16x = f(t)$ represents a spring-mass system without damping having Hooke's constant 16, subject to external force $f(t)$. In a mechanical context, the Dirac delta term $5\delta(t - t_0)$ is an *idealization* of a hammer-hit at time $t = t_0 > 0$ with impulse 5.

More precisely, the forcing term $f(t)$ can be formally written as a Riemann-Stieltjes integrator $5dH(t-t_0)$ where H is Heaviside's unit step function. The Dirac delta or "derivative of the Heaviside unit step," nonsensical as it may appear, is realized in applications via the two-sided or central difference quotient

$$\frac{H(t+h) - H(t-h)}{2h} \approx dH(t).$$

Therefore, the force $f(t)$ in the idealization $5\delta(t-t_0)$ is given for $h>0$ very small by the approximation

$$f(t) \approx 5\frac{H(t-t_0+h)-H(t-t_0-h)}{2h}.$$

The *impulse*[2] of the approximated force over a large interval $[a,b]$ is computed from

$$\int_a^b f(t)dt \approx 5\int_{-h}^{h}\frac{H(t-t_0+h)-H(t-t_0-h)}{2h}dt = 5,$$

due to the integrand being $1/(2h)$ on $|t-t_0|<h$ and otherwise 0.

Modeling Impulses. One argument for the Dirac delta idealization is that an infinity of choices exist for modeling an impulse. There are in addition to the central difference quotient two other popular difference quotients, the forward quotient $(H(t+h)-H(t))/h$ and the backward quotient $(H(t)-H(t-h))/h$ ($h>0$ assumed). In reality, h is unknown in any application, and the impulsive force of a hammer hit is hardly constant, as is supposed by this naive modeling.

The modeling logic often applied for the Dirac delta is that the external force $f(t)$ is used in the model in a limited manner, in which only the momentum $p=mv$ is important. More precisely, only the change in momentum or impulse is important, $\int_a^b f(t)dt = \Delta p = mv(b)-mv(a)$.

The precise force $f(t)$ is replaced during the modeling by a simplistic piecewise-defined force that has exactly the same impulse Δp. The replacement is justified by arguing that if only the impulse is important, and not the actual details of the force, then both models should give similar results.

Function or Operator? The work of physics Nobel prize winner P. Dirac (1902–1984) proceeded for about 20 years before the mathematical community developed a sound mathematical theory for his impulsive force representations. A systematic theory was developed in 1936 by the soviet mathematician S. Sobolev. The French mathematician L. Schwartz further developed the theory in 1945. He observed that the idealization is not a function but an operator or *linear functional*, in particular, δ maps or *associates* to each function $\phi(t)$ its value at $t=0$, in short, $\delta(\phi)=\phi(0)$. This fact was observed early on by Dirac and others, during the replacement of simplistic forces by δ. In Laplace theory, there is a natural encounter with the ideas, because $\mathcal{L}(f(t))$ routinely appears on the right of the equation after transformation. This term, in the case

[2]Momentum is defined to be mass times velocity. If the force f is given by Newton's law as $f(t)=\frac{d}{dt}(mv(t))$ and $v(t)$ is velocity, then $\int_a^b f(t)dt = mv(b)-mv(a)$ is the net momentum or impulse.

of an impulsive force $f(t) = c(H(t-t_0-h)-H(t-t_0+h))/(2h)$, evaluates for $t_0 > 0$ and $t_0 - h > 0$ as follows:

$$\begin{aligned}\mathcal{L}(f(t)) &= \int_0^\infty \frac{c}{2h}(H(t-t_0-h)-H(t-t_0+h))e^{-st}dt \\ &= \int_{t_0-h}^{t_0+h} \frac{c}{2h}e^{-st}dt \\ &= ce^{-st_0}\left(\frac{e^{sh}-e^{-sh}}{2sh}\right)\end{aligned}$$

The factor $\dfrac{e^{sh}-e^{-sh}}{2sh}$ is approximately 1 for $h > 0$ small, because of L'Hospital's rule. The immediate conclusion is that we should replace the impulsive force f by an equivalent one f^* such that

$$\mathcal{L}(f^*(t)) = ce^{-st_0}.$$

Well, *there is no such function* f^*!

The apparent mathematical flaw in this idea was resolved by the work of L. Schwartz on **distributions**. In short, there is a solid foundation for introducing f^*, but unfortunately the mathematics involved is not elementary nor especially accessible to those readers whose background is just calculus.

Practising engineers and scientists might be able to ignore the vast literature on distributions, citing the example of physicist P. Dirac, who succeeded in applying impulsive force ideas without the distribution theory developed by S. Sobolev and L. Schwartz. This will not be the case for those who wish to read current literature on partial differential equations, because the work on distributions has forever changed the required background for that topic.

7.6 Laplace Table Derivations

Verified here are two Laplace tables, the minimal Laplace Table 7.2-4 and its extension Table 7.2-5. Largely, this section is for reading, as it is designed to enrich lectures and to aid readers who study alone.

Derivation of Laplace integral formulas in Table 7.2-4, page 254.

- **Proof of** $\mathcal{L}(t^n) = n!/s^{1+n}$:

The first step is to evaluate $\mathcal{L}(t^n)$ for $n = 0$.

$\mathcal{L}(1) = \int_0^\infty (1)e^{-st}dt$	Laplace integral of $f(t) = 1$.
$= -(1/s)e^{-st}\big\|_{t=0}^{t=\infty}$	Evaluate the integral.
$= 1/s$	Assumed $s > 0$ to evaluate $\lim_{t\to\infty} e^{-st}$.

The value of $\mathcal{L}(t^n)$ for $n = 1$ can be obtained by s-differentiation of the relation $\mathcal{L}(1) = 1/s$, as follows.

$\frac{d}{ds}\mathcal{L}(1) = \frac{d}{ds}\int_0^\infty (1)e^{-st}dt$	Laplace integral for $f(t) = 1$.
$= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right) dt$	Used $\frac{d}{ds}\int_a^b F dt = \int_a^b \frac{dF}{ds} dt$.
$= \int_0^\infty (-t)e^{-st}dt$	Calculus rule $(e^u)' = u'e^u$.
$= -\mathcal{L}(t)$	Definition of $\mathcal{L}(t)$.

Then

$\mathcal{L}(t) = -\frac{d}{ds}\mathcal{L}(1)$	Rewrite last display.
$= -\frac{d}{ds}(1/s)$	Use $\mathcal{L}(1) = 1/s$.
$= 1/s^2$	Differentiate.

This idea can be repeated to give $\mathcal{L}(t^2) = -\frac{d}{ds}\mathcal{L}(t)$ and hence $\mathcal{L}(t^2) = 2/s^3$. The pattern is $\mathcal{L}(t^n) = -\frac{d}{ds}\mathcal{L}(t^{n-1})$ which gives $\mathcal{L}(t^n) = n!/s^{1+n}$.

- **Proof of** $\mathcal{L}(e^{at}) = 1/(s-a)$:

The result follows from $\mathcal{L}(1) = 1/s$, as follows.

$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt$	Direct Laplace transform.
$= \int_0^\infty e^{-(s-a)t}dt$	Use $e^Ae^B = e^{A+B}$.
$= \int_0^\infty e^{-St}dt$	Substitute $S = s - a$.
$= 1/S$	Apply $\mathcal{L}(1) = 1/s$.
$= 1/(s-a)$	Back-substitute $S = s - a$.

- **Proof of** $\mathcal{L}(\cos bt) = s/(s^s + b^2)$ **and** $\mathcal{L}(\sin bt) = b/(s^s + b^2)$:

Use will be made of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, usually first introduced in trigonometry. In this formula, θ is a real number (in radians) and $i = \sqrt{-1}$ is the complex unit.

$e^{ibt}e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$	Substitute $\theta = bt$ into Euler's formula and multiply by e^{-st}.
$\int_0^\infty e^{-ibt}e^{-st}dt = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$	Integrate $t = 0$ to $t = \infty$. Use properties of integrals.
$\dfrac{1}{s-ib} = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$	Evaluate the left side using $\mathcal{L}(e^{at}) = 1/(s-a)$, $a = ib$.
$\dfrac{1}{s-ib} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$	Direct Laplace transform definition.
$\dfrac{s+ib}{s^2+b^2} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$	Use complex rule $1/z = \overline{z}/\lvert z\rvert^2$, $z = A + iB$, $\overline{z} = A - iB$, $\lvert z\rvert = \sqrt{A^2+B^2}$.
$\dfrac{s}{s^2+b^2} = \mathcal{L}(\cos bt)$	Extract the real part.
$\dfrac{b}{s^2+b^2} = \mathcal{L}(\sin bt)$	Extract the imaginary part.

Derivation of Laplace integral formulas in Table 7.2-5, page 254.

- **Proof of the Heaviside formula** $\mathcal{L}(H(t-a)) = e^{-as}/s$.

$\mathcal{L}(H(t-a)) = \int_0^\infty H(t-a)e^{-st}dt$	Direct Laplace transform. Assume $a \ge 0$.
$= \int_a^\infty (1)e^{-st}dt$	Because $H(t-a) = 0$ for $0 \le t < a$.
$= \int_0^\infty (1)e^{-s(x+a)}dx$	Change variables $t = x + a$.
$= e^{-as}\int_0^\infty (1)e^{-sx}dx$	Constant e^{-as} moves outside integral.
$= e^{-as}(1/s)$	Apply $\mathcal{L}(1) = 1/s$.

- **Proof of the Dirac delta formula** $\mathcal{L}(\delta(t-a)) = e^{-as}$.

The *definition* of the delta function is a formal one, in which every occurrence of $\delta(t-a)dt$ under an integrand is replaced by $dH(t-a)$. The differential symbol $dH(t-a)$ is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in [?] for monotonic integrators $\alpha(x)$ as the limit

$$\int_a^b f(x)d\alpha(x) = \lim_{N\to\infty}\sum_{n=1}^{N} f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

where $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \cdots < x_N$ forms a partition of $[a, b]$ whose mesh approaches zero as $N \to \infty$.

The steps in computing the Laplace integral of the delta function appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol $\delta(x)$.

$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)dt$	Laplace integral, $a > 0$ assumed.
$= \int_0^\infty e^{-st}dH(t-a)$	Replace $\delta(t-a)dt$ by $dH(t-a)$.
$= \lim_{M\to\infty}\int_0^M e^{-st}dH(t-a)$	Definition of improper integral.

$= e^{-sa}$ Explained below.

To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$\int_0^M e^{-st} dH(t-a) = \lim_{N\to\infty} \sum_{n=0}^{N-1} e^{-st_n}(H(t_n - a) - H(t_{n-1} - a))$$

where $0 = t_0 < t_1 < \cdots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1\le n\le N}(t_n - t_{n-1})$ approaches zero as $N \to \infty$. Given a partition, if $t_{n-1} < a \le t_n$, then $H(t_n - a) - H(t_{n-1} - a) = 1$, otherwise this factor is zero. Therefore, the sum reduces to a single term e^{-st_n}. This term approaches e^{-sa} as $N \to \infty$, because t_n must approach a.

- **Proof of $\mathcal{L}(\textbf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$:**

The library function **floor** present in computer languages C and Fortran is defined by $\textbf{floor}(x)$ = greatest whole integer $\le x$, e.g., $\textbf{floor}(5.2) = 5$ and $\textbf{floor}(-1.9) = -2$. The computation of the Laplace integral of $\textbf{floor}(t)$ requires ideas from infinite series, as follows.

$F(s) = \int_0^\infty \textbf{floor}(t) e^{-st} dt$	Laplace integral definition.
$= \sum_{n=0}^\infty \int_n^{n+1} (n) e^{-st} dt$	On $n \le t < n+1$, $\textbf{floor}(t) = n$.
$= \sum_{n=0}^\infty \dfrac{n}{s}(e^{-ns} - e^{-ns-s})$	Evaluate each integral.
$= \dfrac{1-e^{-s}}{s} \sum_{n=0}^\infty n e^{-sn}$	Common factor removed.
$= \dfrac{x(1-x)}{s} \sum_{n=0}^\infty n x^{n-1}$	Define $x = e^{-s}$.
$= \dfrac{x(1-x)}{s} \dfrac{d}{dx} \sum_{n=0}^\infty x^n$	Term-by-term differentiation.
$= \dfrac{x(1-x)}{s} \dfrac{d}{dx} \dfrac{1}{1-x}$	Geometric series sum.
$= \dfrac{x}{s(1-x)}$	Compute the derivative, simplify.
$= \dfrac{e^{-s}}{s(1-e^{-s})}$	Substitute $x = e^{-s}$.

To evaluate the Laplace integral of $\textbf{floor}(t/a)$, a change of variables is made.

$\mathcal{L}(\textbf{floor}(t/a)) = \int_0^\infty \textbf{floor}(t/a) e^{-st} dt$	Laplace integral definition.
$= a \int_0^\infty \textbf{floor}(r) e^{-asr} dr$	Change variables $t = ar$.
$= aF(as)$	Apply the formula for $F(s)$.
$= \dfrac{e^{-as}}{s(1-e^{-as})}$	Simplify.

- **Proof of $\mathcal{L}(\textbf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$:**

The square wave defined by $\textbf{sqw}(x) = (-1)^{\textbf{floor}(x)}$ is periodic of period 2 and piecewise-defined. Let $\mathcal{P} = \int_0^2 \textbf{sqw}(t) e^{-st} dt$.

$$\begin{aligned}
\mathcal{P} &= \int_0^1 \mathbf{sqw}(t)e^{-st}dt + \int_1^2 \mathbf{sqw}(t)e^{-st}dt && \text{Apply } \int_a^b = \int_a^c + \int_c^b. \\
&= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt && \text{Use } \mathbf{sqw}(x) = 1 \text{ on } 0 \le x < 1 \text{ and } \mathbf{sqw}(x) = -1 \text{ on } 1 \le x < 2. \\
&= \frac{1}{s}(1 - e^{-s}) + \frac{1}{s}(e^{-2s} - e^{-s}) && \text{Evaluate each integral.} \\
&= \frac{1}{s}(1 - e^{-s})^2 && \text{Collect terms.}
\end{aligned}$$

An intermediate step is to compute the Laplace integral of $\mathbf{sqw}(t)$:

$$\begin{aligned}
\mathcal{L}(\mathbf{sqw}(t)) &= \frac{\int_0^2 \mathbf{sqw}(t)e^{-st}dt}{1 - e^{-2s}} && \text{Periodic function formula, page 277.} \\
&= \frac{1}{s}(1 - e^{-s})^2 \frac{1}{1 - e^{-2s}}. && \text{Use the computation of } \mathcal{P} \text{ above.} \\
&= \frac{1}{s}\frac{1 - e^{-s}}{1 + e^{-s}}. && \text{Factor } 1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s}). \\
&= \frac{1}{s}\frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}. && \text{Multiply the fraction by } e^{s/2}/e^{s/2}. \\
&= \frac{1}{s}\frac{\sinh(s/2)}{\cosh(s/2)}. && \text{Use } \sinh u = (e^u - e^{-u})/2, \ \cosh u = (e^u + e^{-u})/2. \\
&= \frac{1}{s}\tanh(s/2). && \text{Use } \tanh u = \sinh u/\cosh u.
\end{aligned}$$

To complete the computation of $\mathcal{L}(\mathbf{sqw}(t/a))$, a change of variables is made:

$$\begin{aligned}
\mathcal{L}(\mathbf{sqw}(t/a)) &= \int_0^\infty \mathbf{sqw}(t/a)e^{-st}dt && \text{Direct transform.} \\
&= \int_0^\infty \mathbf{sqw}(r)e^{-asr}(a)dr && \text{Change variables } r = t/a. \\
&= \frac{a}{as}\tanh(as/2) && \text{See } \mathcal{L}(\mathbf{sqw}(t)) \text{ above.} \\
&= \frac{1}{s}\tanh(as/2)
\end{aligned}$$

- **Proof of $\mathcal{L}(a\,\mathbf{trw}(t/a)) = \frac{1}{s^2}\tanh(as/2)$:**

The triangular wave is defined by $\mathbf{trw}(t) = \int_0^t \mathbf{sqw}(x)dx$.

$$\begin{aligned}
\mathcal{L}(a\,\mathbf{trw}(t/a)) &= \frac{1}{s}(f(0) + \mathcal{L}(f'(t)) && \text{Let } f(t) = a\,\mathbf{trw}(t/a). \text{ Use } \mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0), \text{ page 253.} \\
&= \frac{1}{s}\mathcal{L}(\mathbf{sqw}(t/a)) && \text{Use } f(0) = 0, \ (a\int_0^{t/a}\mathbf{sqw}(x)dx)' = \mathbf{sqw}(t/a). \\
&= \frac{1}{s^2}\tanh(as/2) && \text{Table entry for } \mathbf{sqw}.
\end{aligned}$$

- **Proof of $\mathcal{L}(t^\alpha) = \frac{\Gamma(1+\alpha)}{s^{1+\alpha}}$:**

$$\begin{aligned}
\mathcal{L}(t^\alpha) &= \int_0^\infty t^\alpha e^{-st}dt && \text{Direct Laplace transform.} \\
&= \int_0^\infty (u/s)^\alpha e^{-u}du/s && \text{Change variables } u = st, \ du = sdt.
\end{aligned}$$

$$= \frac{1}{s^{1+\alpha}} \int_0^\infty u^\alpha e^{-u} du$$

$$= \frac{1}{s^{1+\alpha}} \Gamma(1+\alpha).$$

Where $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$, by definition.

The *generalized factorial function* $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2)\cdots(n)$ in case $x = n+1$ is an integer. In literature, $\alpha!$ means $\Gamma(1+\alpha)$. For more details about the Gamma function, see Abramowitz and Stegun [**?**], or `maple` documentation.

- **Proof of $\mathcal{L}(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$:**

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(1+(-1/2))}{s^{1-1/2}}$$

Apply the previous formula.

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

Use $\Gamma(1/2) = \sqrt{\pi}$.

7.7 Transform Properties

Collected here are the major theorems and their proofs for the manipulation of Laplace transform tables.

Theorem 4 (Linearity)
The Laplace transform has these inherited integral properties:

$$\begin{array}{ll}\textbf{(a)} & \mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)), \\ \textbf{(b)} & \mathcal{L}(cf(t)) = c\mathcal{L}(f(t)).\end{array}$$

Theorem 5 (The t-Derivative Rule)
Let $y(t)$ be continuous, of exponential order and let $f'(t)$ be piecewise continuous on $t \ge 0$. Then $\mathcal{L}(y'(t))$ exists and

$$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0).$$

Theorem 6 (The t-Integral Rule)
Let $g(t)$ be of exponential order and continuous for $t \ge 0$. Then

$$\mathcal{L}\left(\int_0^t g(x)\,dx\right) = \frac{1}{s}\mathcal{L}(g(t)).$$

Theorem 7 (The s-Differentiation Rule)
Let $f(t)$ be of exponential order. Then

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t)).$$

Theorem 8 (First Shifting Rule)
Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))|_{s\to(s-a)}\,.$$

Theorem 9 (Second Shifting Rule)
Let $f(t)$ and $g(t)$ be of exponential order and assume $a \ge 0$. Then

$$\begin{array}{ll}\textbf{(a)} & \mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t)), \\ \textbf{(b)} & \mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a)).\end{array}$$

Theorem 10 (Periodic Function Rule)
Let $f(t)$ be of exponential order and satisfy $f(t+P) = f(t)$. Then

$$\mathcal{L}(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1 - e^{-Ps}}.$$

Theorem 11 (Convolution Rule)
Let $f(t)$ and $g(t)$ be of exponential order. Then

$$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(x)g(t-x)dx\right).$$

Proof of Theorem 4 (linearity):

LHS $= \mathcal{L}(f(t)+g(t))$	Left side of the identity in (a).
$= \int_0^\infty (f(t)+g(t))e^{-st}dt$	Direct transform.
$= \int_0^\infty f(t)e^{-st}dt + \int_0^\infty g(t)e^{-st}dt$	Calculus integral rule.
$= \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	Equals RHS; identity (a) verified.
LHS $= \mathcal{L}(cf(t))$	Left side of the identity in (b).
$= \int_0^\infty cf(t)e^{-st}dt$	Direct transform.
$= c\int_0^\infty f(t)e^{-st}dt$	Calculus integral rule.
$= c\mathcal{L}(f(t))$	Equals RHS; identity (b) verified.

Proof of Theorem 5 (t-derivative rule): Already $\mathcal{L}(f(t))$ exists, because f is of exponential order and continuous. On an interval $[a,b]$ where f' is continuous, integration by parts using $u = e^{-st}$, $dv = f'(t)dt$ gives

$$\int_a^b f'(t)e^{-st}dt = f(t)e^{-st}\Big|_{t=a}^{t=b} - \int_a^b f(t)(-s)e^{-st}dt$$
$$= -f(a)e^{-sa} + f(b)e^{-sb} + s\int_a^b f(t)e^{-st}dt.$$

On any interval $[0,N]$, there are finitely many intervals $[a,b]$ on each of which f' is continuous. Add the above equality across these finitely many intervals $[a,b]$. The boundary values on adjacent intervals match and the integrals add to give

$$\int_0^N f'(t)e^{-st}dt = -f(0)e^0 + f(N)e^{-sN} + s\int_0^N f(t)e^{-st}dt.$$

Take the limit across this equality as $N \to \infty$. Then the right side has limit $-f(0) + s\mathcal{L}(f(t))$, because of the existence of $\mathcal{L}(f(t))$ and $\lim_{t\to\infty} f(t)e^{-st} = 0$ for large s. Therefore, the left side has a limit, and by definition $\mathcal{L}(f'(t))$ exists and $\mathcal{L}(f'(t)) = -f(0) + s\mathcal{L}(f(t))$.

Proof of Theorem 6 (t-Integral rule): Let $f(t) = \int_0^t g(x)dx$. Then f is of exponential order and continuous. The details:

$\mathcal{L}(\int_0^t g(x)dx) = \mathcal{L}(f(t))$	By definition.
$= \frac{1}{s}\mathcal{L}(f'(t))$	Because $f(0) = 0$ implies $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t))$.
$= \frac{1}{s}\mathcal{L}(g(t))$	Because $f' = g$ by the Fundamental theorem of calculus.

Proof of Theorem 7 (s-differentiation): We prove the equivalent relation $\mathcal{L}((-t)f(t)) = (d/ds)\mathcal{L}(f(t))$. If f is of exponential order, then so is $(-t)f(t)$, therefore $\mathcal{L}((-t)f(t))$ exists. It remains to show the s-derivative exists and satisfies the given equality.

The proof below is based in part upon the calculus inequality

$$(1)\qquad \left|e^{-x} + x - 1\right| \le x^2, \quad x \ge 0.$$

The inequality is obtained from two applications of the *mean value theorem* $g(b)-g(a)=g'(\overline{x})(b-a)$, which gives $e^{-x}+x-1 = x\overline{x}e^{-x_1}$ with $0 \le x_1 \le \overline{x} \le x$.

In addition, the existence of $\mathcal{L}(t^2|f(t)|)$ is used to define $s_0 > 0$ such that $\mathcal{L}(t^2|f(t)|) \le 1$ for $s > s_0$. This follows from the transform existence theorem for functions of exponential order, where it is shown that the transform has limit zero at $s = \infty$.

Consider $h \ne 0$ and the Newton quotient $Q(s,h) = (F(s+h)-F(s))/h$ for the s-derivative of the Laplace integral. We have to show that

$$\lim_{h\to 0} |Q(s,h) - \mathcal{L}((-t)f(t))| = 0.$$

This will be accomplished by proving for $s > s_0$ and $s+h > s_0$ the inequality

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h|.$$

For $h \ne 0$,

$$Q(s,h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)\frac{e^{-st-ht} - e^{-st} + the^{-st}}{h}\,dt.$$

Assume $h > 0$. Due to the exponential rule $e^{A+B} = e^Ae^B$, the quotient in the integrand simplifies to give

$$Q(s,h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)e^{-st}\left(\frac{e^{-ht} + th - 1}{h}\right)dt.$$

Inequality (1) applies with $x = ht \ge 0$, giving

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h|\int_0^\infty t^2|f(t)|e^{-st}dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$, which for $s > s_0$ is bounded by $|h|$, completing the proof for $h > 0$. If $h < 0$, then a similar calculation is made to obtain

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h|\int_0^\infty t^2|f(t)e^{-st-ht}dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$ evaluated at $s+h$ instead of s. If $s+h > s_0$, then the right side is bounded by $|h|$, completing the proof for $h < 0$.

Proof of Theorem 8 (first shifting rule): The left side LHS of the equality can be written because of the exponential rule $e^Ae^B = e^{A+B}$ as

$$\text{LHS} = \int_0^\infty f(t)e^{-(s-a)t}dt.$$

This integral is $\mathcal{L}(f(t))$ with s replaced by $s-a$, which is precisely the meaning of the right side RHS of the equality. Therefore, LHS = RHS.

Proof of Theorem 9 (second shifting rule): The details for (a) are

$$\begin{aligned}\text{LHS} &= \mathcal{L}(H(t-a)f(t-a)) \\ &= \textstyle\int_0^\infty H(t-a)f(t-a)e^{-st}dt \quad \text{Direct transform.}\end{aligned}$$

$= \int_a^\infty H(t-a)f(t-a)e^{-st}dt$	Because $a \geq 0$ and $H(x) = 0$ for $x < 0$.
$= \int_0^\infty H(x)f(x)e^{-s(x+a)}dx$	Change variables $x = t - a$, $dx = dt$.
$= e^{-sa}\int_0^\infty f(x)e^{-sx}dx$	Use $H(x) = 1$ for $x \geq 0$.
$= e^{-sa}\mathcal{L}(f(t))$	Direct transform.
$=$ RHS	Identity (a) verified.

In the details for (b), let $f(t) = g(t+a)$, then

LHS $= \mathcal{L}(H(t-a)g(t))$	
$= \mathcal{L}(H(t-a)f(t-a))$	Use $f(t-a) = g(t-a+a) = g(t)$.
$= e^{-sa}\mathcal{L}(f(t))$	Apply (a).
$= e^{-sa}\mathcal{L}(g(t+a))$	Because $f(t) = g(t+a)$.
$=$ RHS	Identity (b) verified.

Proof of Theorem 10 (periodic function rule):

LHS $= \mathcal{L}(f(t))$	
$= \int_0^\infty f(t)e^{-st}dt$	Direct transform.
$= \sum_{n=0}^\infty \int_{nP}^{nP+P} f(t)e^{-st}dt$	Additivity of the integral.
$= \sum_{n=0}^\infty \int_0^P f(x+nP)e^{-sx-nPs}dx$	Change variables $t = x + nP$.
$= \sum_{n=0}^\infty e^{-nPs}\int_0^P f(x)e^{-sx}dx$	Because f is P-periodic and $e^A e^B = e^{A+B}$.
$= \int_0^P f(x)e^{-sx}dx \sum_{n=0}^\infty r^n$	Common factor in summation. Define $r = e^{-Ps}$.
$= \int_0^P f(x)e^{-sx}dx \dfrac{1}{1-r}$	Sum the geometric series.
$= \dfrac{\int_0^P f(x)e^{-sx}dx}{1-e^{-Ps}}$	Substitute $r = e^{-Ps}$.
$=$ RHS	Periodic function identity verified.

Left unmentioned here is the convergence of the infinite series on line 3 of the proof, which follows from f of exponential order.

Proof of Theorem 11 (convolution rule): The details use Fubini's integration interchange theorem for a planar unbounded region, and therefore this proof involves advanced calculus methods that may be outside the background of the reader. Modern calculus texts contain a less general version of Fubini's theorem for finite regions, usually referenced as *iterated integrals*. The unbounded planar region is written in two ways:

$$D = \{(r,t) : t \leq r < \infty,\ 0 \leq t < \infty\},$$
$$\mathcal{D} = \{(r,t) : 0 \leq r < \infty,\ 0 \leq r \leq t\}.$$

Readers should pause here and verify that $D = \mathcal{D}$.

The change of variable $r = x + t$, $dr = dx$ is applied for fixed $t \geq 0$ to obtain the identity

$$(2) \qquad \begin{aligned} e^{-st}\int_0^\infty g(x)e^{-sx}dx &= \int_0^\infty g(x)e^{-sx-st}dx \\ &= \int_t^\infty g(r-t)e^{-rs}dr. \end{aligned}$$

The left side of the convolution identity is expanded as follows:

$$\begin{aligned}
\mathsf{LHS} &= \mathcal{L}(f(t))\mathcal{L}(g(t)) & \\
&= \int_0^\infty f(t)e^{-st}dt \int_0^\infty g(x)e^{-sx}dx & \text{Direct transform.} \\
&= \int_0^\infty f(t)\int_t^\infty g(r-t)e^{-rs}drdt & \text{Apply identity (2).} \\
&= \int_D f(t)g(r-t)e^{-rs}drdt & \text{Fubini's theorem applied.} \\
&= \int_{\mathcal{D}} f(t)g(r-t)e^{-rs}drdt & \text{Descriptions } D \text{ and } \mathcal{D} \text{ are the same.} \\
&= \int_0^\infty \int_0^r f(t)g(r-t)dte^{-rs}dr & \text{Fubini's theorem applied.}
\end{aligned}$$

Then

$$\begin{aligned}
\mathsf{RHS} &= \mathcal{L}\left(\int_0^t f(u)g(t-u)du\right) & \\
&= \int_0^\infty \int_0^t f(u)g(t-u)due^{-st}dt & \text{Direct transform.} \\
&= \int_0^\infty \int_0^r f(u)g(r-u)due^{-sr}dr & \text{Change variable names } r \leftrightarrow t. \\
&= \int_0^\infty \int_0^r f(t)g(r-t)dt\, e^{-sr}dr & \text{Change variable names } u \leftrightarrow t. \\
&= \mathsf{LHS} & \text{Convolution identity verified.}
\end{aligned}$$