Vector space V

It is a **data set** V. Storage uses some organization system. Included is a **toolkit** of eight (8) algebraic properties.

Closure	The operations $\vec{X}+\vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V.	
Addition	$\vec{X}+\vec{Y}=\vec{Y}+\vec{X}$	commutative
	$\vec{X}+(\vec{Y}+\vec{Z})=(\vec{Y}+\vec{X})+\vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0}+\vec{X}=\vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X}+(-\vec{X})=\vec{0}$	negative
Scalar multiply	$k(\vec{X}+\vec{Y})=k\vec{X}+k\vec{Y}$	distributive I
	$(k_1+k_2)\vec{X}=k_1\vec{X}+k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X})=(k_1k_2)\vec{X}$	distributive III
	$1\vec{X}=\vec{X}$	identity

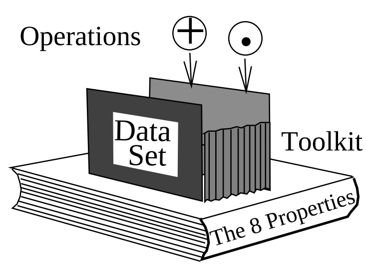

Figure 3. A Data Storage System.
A vector space is a data set storage system which organizes data. The data set is equipped with a toolkit consisting of operations $+$ and $\cdot$ plus 8 algebraic vector space properties.

Theorem 5 (Subspaces and Restriction Equations) Let V be one of the vector spaces R^n and let A be an $m \times n$ matrix. Define a smaller set of data items from V by the equation

$$S = \{\mathbf{x} \ : \ \mathbf{x} \text{ in } S, \quad A\mathbf{x} = \mathbf{0}\}.$$

Then S is a subspace of V, that is, operations of addition and scalar multiplication applied to data items in S give answers in S and the 8-property toolkit applies to data items in S.

Proof: Zero is in V because $A\mathbf{0} = \mathbf{0}$ for any matrix A. To verify the subspace criterion, we verify that $\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y}$ for $\mathbf{x}$ and $\mathbf{y}$ in V also belongs to V. The details:

$A\mathbf{z} = A(c_1\mathbf{x} + c_2\mathbf{y})$

$= A(c_1\mathbf{x}) + A(c_2\mathbf{y})$

$= c_1 A\mathbf{x} + c_2 A\mathbf{y}$

$= c_1\mathbf{0} + c_2\mathbf{0}$ Because $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, due to $\mathbf{x}$, $\mathbf{y}$ in V.

$= \mathbf{0}$ Therefore, $A\mathbf{z} = \mathbf{0}$, and $\mathbf{z}$ is in V.

The proof is complete.

Independence test for two vectors $\mathbf{v}_1$, $\mathbf{v}_2$.

In an abstract vector space V, form the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

Solve this equation for c_1, c_2. Then $\mathbf{v}_1$, $\mathbf{v}_2$ are independent in V only if the system has unique solution $c_1 = c_2 = 0$.

Illustration. Two column vectors are tested for independence by forming the system of equations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, e.g,

$$c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a homogeneous system $A\mathbf{c} = \mathbf{0}$ with

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The system $A\mathbf{c} = \mathbf{0}$ can be solved for $\mathbf{c}$ by $\mathbf{rref}$ methods. Because $\mathbf{rref}(A) = I$, then $c_1 = c_2 = 0$, which verifies independence.

If the system $A\mathbf{c} = \mathbf{0}$ is square, then $\det(A) \neq 0$ applies to test independence. There is **no chance to use determinants** when the system is not square. For instance, in R^3, the homogeneous system

$$c_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has vector-matrix form $A\mathbf{c} = \mathbf{0}$ with 3×2 matrix A.

Rank Test.

In the vector space R^n, the key to detection of independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. The test is justified from the formula $\mathbf{nullity}(A) + \mathbf{rank}(A) = k$, where k is the column dimension of A.

Theorem 6 (Rank-Nullity Test)
Let $\mathbf{v}_1$, ..., $\mathbf{v}_k$ be k column vectors in R^n and let A be the augmented matrix of these vectors. The vectors are independent if $\mathbf{rank}(A) = k$ and dependent if $\mathbf{rank}(A) < k$. The conditions are equivalent to $\mathbf{nullity}(A) = 0$ and $\mathbf{nullity}(A) > 0$, respectively.

Determinant Test.

In the unusual case when the system arising in the independence test can be expressed as $A\mathbf{c} = \mathbf{0}$ and A is square, then $\det(A) = 0$ detects dependence, and $\det(A) \neq 0$ detects independence. The reasoning is based upon the formula $A^{-1} = \textbf{adj}(A)/\det(A)$, valid exactly when $\det(A) \neq 0$.

Theorem 7 (Determinant Test)
Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n column vectors in R^n and let A be the augmented matrix of these vectors. The vectors are independent if $\det(A) \neq 0$ and dependent if $\det(A) = 0$.

The following test enumerates three common conditions for which S fails to pass the sanity test for a subspace. It is justified from the subspace criterion.

Theorem 8 (Testing S not a Subspace)
Let V be an abstract vector space and assume S is a subset of V. Then S is not a subspace of V provided one of the following holds.

(1) The vector 0 is not in S.

(2) Some $\mathbf{x}$ and $-\mathbf{x}$ are not both in S.

(3) Vector $\mathbf{x} + \mathbf{y}$ is not in S for some $\mathbf{x}$ and $\mathbf{y}$ in S.