Chapter 6

Topics in Linear Differential Equations

Developed here is the theory for higher order linear constant-coefficient differential equations. Besides a basic recipe for the solution of such equations, extensions are developed for the topics of variation of parameters and undetermined coefficients.

Enrichment topics include the Cauchy-Euler differential equation, the Cauchy kernel for second order linear differential equations, and a library of special methods for undetermined coefficients methods, the latter having prerequisites of only basic calculus and college algebra. Developed with the library methods is a verification of the method of undetermined coefficients, via Kümmer's method.

6.1 Higher Order Linear Equations

Developed here is the recipe for higher order linear differential equations with constant coefficients

$$y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0. \tag{1}$$

The variation of parameters formula and the method of undetermined coefficients are discussed for the associated forced equation

$$y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(x). \tag{2}$$

A Recipe for Higher Order Equations

Consider equation (1) with **real** coefficients. The **characteristic equation** of (1) is the polynomial equation

$$r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0. \tag{3}$$

The general solution y of (1) is constructed as follows.

Higher Order Recipe Stage 1.

Repeat (I) below for all distinct real roots $r = a$ of the characteristic equation (3). Symbol k is the maximum power such that $(r-a)^k$ divides the characteristic polynomial, which means that k equals the algebraic multiplicity of the root $r = a$.

(I) The equation $r - a = 0$ is the characteristic equation of $u' - au = 0$, having general solution

$$u = u_0 e^{ax}.$$

Replace u_0 by a polynomial in x with k arbitrary coefficients. Add the modified expression u to the general solution y.

Higher Order Recipe Stage 2.

Repeat (II) below for all distinct complex roots $z = a + ib$, $b > 0$, of the characteristic equation (3). Symbol k is the maximum power such that $(r - z)^k$ divides the characteristic polynomial, which means that k equals the algebraic multiplicity of the root $r = z$.

(II) The equation $(r - z)(r - \overline{z}) = 0$ is the characteristic equation of a second order differential equation whose **Case 3** recipe solution is

$$u = u_1 e^{ax} \cos bx + u_2 e^{ax} \sin bx.$$

Replace the constants u_1, u_2 by polynomials in x with k arbitrary coefficients, a total of $2k$ coefficients. Add the modified expression u to the general solution y.

Exponential Solutions. Characteristic equation (3) is formally obtained from the differential equation by replacing $y^{(k)}$ by r^k. This device for remembering how to form the characteristic equation is attributed to **Euler**, because of the following fact.

Theorem 1 (Euler's Exponential Substitution)
Let w be a real or complex number. The function $y(x) = e^{wx}$ is a solution of (1) if and only if $r = w$ is a root of the characteristic equation (3).

Factorization. According to the fundamental theorem of algebra, equation (3) has exactly n roots, counted according to multiplicity. Some number of the roots are real and the remaining roots appear in complex conjugate pairs. This implies that every characteristic equation has a **factored form**

$$(r - a_1)^{k_1} \cdots (r - a_q)^{k_q} Q_1(r)^{m_1} \cdots Q_p(r)^{m_p} = 0$$

where $a_1, \ldots, a_q$ are the **distinct real roots** of the characteristic equation of algebraic multiplicities $k_1, \ldots, k_q$, respectively, and $Q_1(r), \ldots, Q_p(r)$ are the distinct real quadratic factors of the form $(r-z)(r-\overline{z})$, where z exhausts the **distinct complex roots** $z = a + ib$ with $b > 0$, having corresponding multiplicities $m_1, \ldots, m_p$.

Some Recipe Details. Recipe **Stage 1** loops on the distinct linear factors while recipe **Stage 2** loops on the distinct real quadratic factors. The y-differential equation can be expressed in D-operator notation as

$$\left((D-a_1)^{k_1} \cdots (D-a_q)^{k_q} Q_1(D)^{m_1} \cdots Q_p(D)^{m_p}\right) y = 0.$$

The recipe is based upon the fact that the general solution y is the sum of general solution expressions obtained from each distinct factor in this operator form. Specifically, the general solution of

$$(D-a)^{k+1} y = 0$$

is a polynomial $u = c_0 + c_1 x + \cdots + c_k x^k$ with $k+1$ terms times e^{ax}. This fact is proved by the change of variable $y = e^{ax} u$, which finds an equivalent equation $D^{k+1} u = 0$, solvable by quadrature.

An Illustration of the Higher Order Recipe.

Consider the problem of solving a constant coefficient linear differential equation (1) of order 11 having factored characteristic equation

$$(r-2)^3 (r+1)^2 (r^2+4)^2 (r^2+4r+5) = 0.$$

To be applied is the recipe for higher order equations. Then **Stage 1** loops on the two linear factors $r-2$ and $r+1$, while **Stage 2** loops on the two real quadratic factors r^2+4 and r^2+4r+5.

Hand solutions can be organized by a tabular method for generating the general solution y.

Factor	$(r-2)^3$	$(r+1)^2$	$(r^2+4)^2$	(r^2+4r+5)
Multiplicity	3	2	2	1
Base Root	$r=2$	$r=-1$	$r=0+2i$	$r=-2+i$
Base Solution	$u_0 e^{2x}$	$u_0^* e^{-x}$	$u_1 \cos 2x$ $+u_2 \sin 2x$	$u_1^* e^{-2x} \cos x$ $+u_2^* e^{-2x} \sin x$

Symbols $c_1, \ldots, c_{11}$ will represent arbitrary constants in the general solution y. Symbols u_0, u_0^*, u_1, u_2, u_1^*, u_2^* initially represent constants, but they will be assigned polynomial expressions, according to root multiplicity, as follows.

Root	Multiplicity	Polynomial Assigned
$r = 2$	3	$u_0 = c_1 + c_2x + c_3x^2$
$r = -1$	2	$u_0^* = c_4 + c_5x$
$r = 0 + 2i$	2	$u_1 = c_6 + c_7x$
		$u_2 = c_8 + c_9x$
$r = -2 + i$	1	$u_1^* = c_{10}$
		$u_2^* = c_{11}$

The recipe **Stage 1** and **Stage 2** solutions are added to y, giving

$$\begin{aligned} y &= u_0e^{2x} + u_0^*e^{-x} + u_1\cos 2x + u_2\sin 2x \\ &\quad +u_1^*e^{-2x}\cos x + u_2^*e^{-2x}\sin x \\ &= (c_1 + c_2x + c_3x^2)e^{2x} \\ &\quad +(c_4 + c_5x)e^{-x} \\ &\quad +(c_6 + c_7x)\cos 2x + (c_8 + c_9x)\sin 2x \\ &\quad +c_{10}e^{-2x}\cos x + c_{11}e^{-2x}\sin x. \end{aligned}$$

Computer Algebra System Solution. The system `maple` can symbolically solve a higher order equation. Below, `@` is the function composition operator, `@@` is the repeated composition operator and `D` is the differentiation operator. The coding writes the factors of $(r-2)^3(r+1)^2(D^2+4)^2(D^2+4D+5)$ as differential operators $(D-2)^3$, $(D+1)^2$, $(D^2+4)^2$, D^2+4D+5. Then the differential equation is the composition of the component factors.

```
id:=x->x;
F1:=(D-2*id) @@ 3;
F2:=(D+id) @@ 2;
F3:=(D@D+4*id) @@ 2;
F4:=D@D+4*D+5*id;
de:=(F1@F2@F3@F4)(y)(x)=0:
dsolve({de},y(x));
```

Variation of Parameters Formula

The Picard-Lindelöf theorem implies a unique solution defined on $(-\infty, \infty)$ for the initial value problem

$$\begin{aligned} &y^n + a_{n-1}y^{(n-1)} + \cdots + a_0y = 0, \\ &y(0) = \cdots = y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1. \end{aligned} \tag{4}$$

The unique solution is called **Cauchy's kernel**, written $\mathcal{K}(x)$.

To illustrate, Cauchy's kernel $\mathcal{K}(x)$ for $y''' - y'' = 0$ is obtained from its general solution $y = c_1 + c_2x + c_3e^x$ by computing the values of the constants from initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, giving $\mathcal{K}(x) = e^x - x - 1$.

Theorem 2 (Higher Order Variation of Parameters)
Let $y^n + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(x)$ have constant coefficients $a_0, \ldots, a_{n-1}$ and continuous forcing term $r(x)$. Denote by $\mathcal{K}(x)$ Cauchy's kernel for the homogeneous differential equation. Then a particular solution is given by the **variation of parameters formula**

$$y_p(x) = \int_0^x \mathcal{K}(x-u)r(u)du. \tag{5}$$

This solution has zero initial conditions $y(0) = \cdots = y^{(n-1)}(0) = 0$.

Proof: Define $y(x) = \int_0^x \mathcal{K}(x-u)r(u)du$. Compute by the 2-variable chain rule applied to $F(x,y) = \int_0^x \mathcal{K}(y-u)r(u)du$ the formulae

$$\begin{aligned} y(x) &= F(x,x) \\ &= \textstyle\int_0^x \mathcal{K}(x-u)r(u)du, \\ y'(x) &= F_x(x,x,) + F_y(x,x) \\ &= \mathcal{K}(x-x)r(x) + \textstyle\int_0^x \mathcal{K}'(x-u)r(u)du \\ &= 0 + \textstyle\int_0^x \mathcal{K}'(x-u)r(u)du. \end{aligned}$$

The process can be continued to obtain for $0 \le p < n-1$ the general relation

$$y^{(p)}(x) = \int_0^x \mathcal{K}^{(p)}r(u)du.$$

The relation justifies the initial conditions $y(0) = \cdots = y^{(n-1)}(0) = 0$, because each integral is zero at $x = 0$. Take $p = n-1$ and differentiate once again to give

$$y^{(n)}(x) = \mathcal{K}^{(n-1)}(x-x)r(x) + \int_0^x \mathcal{K}^{(n)}r(u)du.$$

Because $\mathcal{K}^{(n-1)}(0) = 1$, this relation implies

$$y^{(n)} + \sum_{p=0}^{n-1} a_p y^{(p)} = r(x) + \int_0^x \left(\mathcal{K}^{(n)}(x-u) + \sum_{p=0}^{n-1} a_p \mathcal{K}^{(p)}(x-u)\right) r(u)du.$$

The sum under the integrand on the right is zero, because Cauchy's kernel satisfies the homogeneous differential equation. This proves $y(x)$ satisfies the nonhomogeneous differential equation. The proof is complete.

Undetermined Coefficients Method

The method applies to higher order nonhomogeneous differential equations

$$y' + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(x). \tag{6}$$

It finds a particular solution y_p of (6) *without* the integration steps present in variation of parameters. The requirements and limitations:

1. The coefficients on the left side of (6) are constant.
2. The function $r(x)$ is a sum of constants times atoms.

An **atom** is a term having one of the forms

$$x^m,\ x^m e^{ax},\ x^m \cos bx,\ x^m \sin bx,\ x^m e^{ax} \cos bx \quad \text{or} \quad x^m e^{ax} \sin bx.$$

The symbols a and b are real constants, with $b > 0$. Symbol $m \ge 0$ is an integer. Atoms A and B are called **related atoms** if their successive derivative formulae contain a common atom.

Higher Order Basic Trial Solution Method

1. Repeatedly differentiate the atoms of $r(x)$ until no new atoms appear. Multiply the distinct atoms so found by **undetermined coefficients** $d_1, d_2, \ldots, d_k$, then add to define a **trial solution** y.
2. **Fixup rule**: if the homogeneous equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$ has solution y_h containing an atom A which appears in the trial solution y, then replace each **related atom** B in y by xB (other atoms appearing in y are unchanged). Repeat the fixup rule until y contains no atom of y_h. The modified expression y is called the **corrected trial solution**.
3. Substitute y into the differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = r(x)$. Match atoms left and right to write out linear algebraic equations for the undetermined coefficients $d_1, d_2, \ldots, d_k$.
4. Solve the equations. The trial solution y with evaluated coefficients $d_1, d_2, \ldots, d_k$ becomes the particular solution y_p.

Higher Order Undetermined Coefficients Illustration.

We will solve

$$y''' - y'' = xe^x + 2x + 1 + 3\sin x,$$

verifying

$$y_p(x) = -\frac{3}{2}x^2 - \frac{1}{3}x^3 - 2xe^x + \frac{1}{2}x^2e^x + \frac{3}{2}\cos x + \frac{3}{2}\sin x.$$

Solution:

Test Applicability. The right side $r(x) = xe^x + 2x + 1 + 3\sin x$ is a sum of terms constructed from the atoms xe^x, x, 1, $\sin x$. The left side has constant coefficients. Therefore, the method of undetermined coefficients applies to find a particular solution y_p.

Trial Solution. The atoms of $r(x)$ are subjected to differentiation. The distinct atoms so found are 1, x, e^x, xe^x, $\cos x$, $\sin x$ (drop coefficients to identify new atoms). The initial trial solution is the expression

$$y = d_1(1) + d_2(x) + d_3(e^x) + d_4(xe^x) + d_5(\cos x) + d_6(\sin x).$$

The general solution $y_h = c_1 + c_2x + c_3e^x$ of $y''' - y'' = 0$ has atoms 1, x, e^x, all of which appear in the trial solution y. Multiply related atoms 1, x in y by x^2 to eliminate duplicate atoms 1, x which appear in y_h. Then multiply related atoms e^x, xe^x in y by x to eliminate the duplicate atom e^x which appears in y_h. The other atoms $\cos x$, $\sin x$ in y are unaffected by the fixup rule, because they are unrelated to atoms of y_h. The final trial solution is

$$y = d_1(x^2) + d_2(x^3) + d_3(xe^x) + d_4(x^2e^x) + d_5(\cos x) + d_6(\sin x).$$

Equations. To substitute the trial solution y into $y''' - y''$ requires formulae for y', y'', y''':

$$\begin{aligned}
y' &= 2\,d_1x + 3\,d_2x^2 + d_3e^xx + d_3e^x + 2\,d_4xe^x + d_4x^2e^x \\
&\quad - d_5\sin(x) + d_6\cos(x), \\
y'' &= 2\,d_1 + 6\,d_2x + d_3e^xx + 2\,d_3e^x + 2\,d_4e^x + 4\,d_4xe^x + d_4x^2e^x \\
&\quad - d_5\cos(x) - d_6\sin(x), \\
y''' &= 6\,d_2 + d_3e^xx + 3\,d_3e^x + 6\,d_4e^x + 6\,d_4xe^x + d_4x^2e^x \\
&\quad + d_5\sin(x) - d_6\cos(x)
\end{aligned}$$

Then

$$\begin{aligned}
r(x) &= y''' - y'' && \text{The given equation.} \\
&= 6d_2 - 2d_1 - 6d_2x + (d_3 + 4d_4)e^x + 2d_4xe^x && \text{Substitute, then} \\
&\quad + (d_5 - d_6)\cos(x) + (d_5 + d_6)\sin(x) && \text{collect like terms.}
\end{aligned}$$

Also, $r(x) \equiv 1 + 2x + xe^x + 3\sin x$. Coefficients of atoms on the left and right must match. Writing out the matches gives the equations

$$\begin{aligned}
-2d_1 + 6d_2 &= 1, \\
-6d_2 &= 2, \\
d_3 + 4d_4 &= 0, \\
2d_4 &= 1, \\
d_5 - d_6 &= 0, \\
d_5 + d_6 &= 3.
\end{aligned}$$

Solve. The first four equations can be solved by back-substitution to give $d_2 = -1/3$, $d_1 = -3/2$, $d_4 = 1/2$, $d_3 = -2$. The last two equations are solved by elimination or Cramer's rule to give $d_5 = 3/2$, $d_6 = 3/2$.

Report y_p. The trial solution y with evaluated coefficients $d_1, \ldots, d_6$ becomes

$$y_p(x) = -\frac{3}{2}x^2 - \frac{1}{3}x^3 - 2xe^x + \frac{1}{2}x^2e^x + \frac{3}{2}\cos x + \frac{3}{2}\sin x.$$

Exercises 6.1

Higher Order Recipe Factored.

Solve the higher order equation with the given characteristic equation. Use the higher order recipe and display a table of distinct roots, multiplicities and base solutions. Verify the general solution y with a computer algebra system, if possible.

1. $(r-1)(r+2)(r-3)^2 = 0$

2. $(r-1)^2(r+2)(r+3) = 0$

3. $(r-1)^3(r+2)^2r^4 = 0$

4. $(r-1)^2(r+2)^3r^5 = 0$

5. $r^2(r-1)^2(r^2+4r+6) = 0$

6. $r^3(r-1)(r^2+4r+6)^2 = 0$

7. $(r-1)(r+2)(r^2+1)^2 = 0$

8. $(r-1)^2(r+2)(r^2+1) = 0$

9. $(r-1)^3(r+2)^2(r^2+4) = 0$

10. $(r-1)^4(r+2)(r^2+4)^2 = 0$

Higher Order Recipe Unfactored.

Completely factor the given characteristic equation, then report the general solution according to the higher order recipe. Check the answer in a computer algebra system, if possible.

11. $(r-1)(r^2-1)^2(r^2+1)^3 = 0$

12. $(r+1)^2(r^2-1)^2(r^2+1)^2 = 0$

13. $(r+2)^2(r^2-4)^2(r^2+16)^2 = 0$

14. $(r+2)^3(r^2-4)^4(r^2+5)^2 = 0$

15. $(r^3-1)^2(r-1)^2(r^2-1) = 0$

16. $(r^3-8)^2(r-2)^2(r^2-4) = 0$

17. $(r^2-4)^3(r^4-16)^2 = 0$

18. $(r^2+8)(r^4-64)^2 = 0$

19. $(r^2-r+1)(r^3+1)^2 = 0$

20. $(r^2+r+1)^2(r^3-1) = 0$

Atoms and Higher Order Equations.

21. Explain why the derivatives of atom x^3e^x satisfy a higher order equation with characteristic equation $(r-1)^4 = 0$.

22. Explain why the derivatives of atom $x^3 \sin x$ satisfy a higher order equation with characteristic equation $(r^2+1)^4 = 0$.

23. Consider a fourth order equation with characteristic equation $(r-a)^4 = 0$ and general solution y. Define $y = ue^{ax}$. Find the differential equation for u.

24. A polynomial $u = c_0 + c_1x + c_2x^2$ satisfies $u''' = 0$. Define $y = ue^{ax}$. Prove that y satisfies a third order equation and determine its characteristic equation.

25. Let y be a solution of a higher order constant-coefficient linear equation. Prove that the derivatives of y satisfy the same differential equation.

26. Let y be a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 = 0$. Explain why y''' is a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5r^3 = 0$.

27. Let atom $A = x^2\cos x$ appear in the general solution of a linear higher order equation. What atoms related to A must also appear in the general solution?

28. Let atom $A = xe^x\cos 2x$ appear in the general solution of a linear higher order equation. What atoms related to A must also appear in the general solution?

29. Let a higher order equation have characteristic equation $(r-9)^3(r-5)^2(r^2+4)^5 = 0$. Explain why the general solution is a sum of constants times atoms.

30. Explain why a higher order equation has general solution a sum of constants times atoms.

Variation of Parameters. Solve the higher order equation given by its characteristic equation and right side $r(x)$. Display the Cauchy kernel $\mathcal{K}(x)$ and a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to evaluate integrals, if possible.

31. $(r-1)(r+2)(r-3)^2 = 0$, $r(x) = e^x$

32. $(r-1)^2(r+2)(r+3) = 0$, $r(x) = e^x$

33. $(r-1)^3(r+2)^2 r^4 = 0$, $r(x) = x + e^{-2x}$

34. $(r-1)^2(r+2)^3 r^5 = 0$, $r(x) = x + e^{-2x}$

35. $r^2(r-1)^2(r^2+4r+6) = 0$, $r(x) = x + e^x$

36. $r^3(r-1)(r^2+4r+6)^2 = 0$, $r(x) = x^2 + e^x$

37. $(r-1)(r+2)(r^2+1)^2 = 0$, $r(x) = \cos x + e^{-2x}$

38. $(r-1)^2(r+2)(r^2+1) = 0$, $r(x) = \sin x + e^{-2x}$

39. $(r-1)^3(r+2)^2(r^2+4) = 0$, $r(x) = \cos 2x + e^x$

40. $(r-1)^4(r+2)(r^2+4)^2 = 0$, $r(x) = \sin 2x + e^x$

Undetermined Coefficient Method. A higher order equation is given by its characteristic equation and right side $r(x)$. Display (a) a trial solution, (b) a system of equations for the undetermined coefficients, and (c) a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to solve for undetermined coefficients, if possible.

41. $(r-1)(r+2)(r-3)^2 = 0$, $r(x) = e^x$

42. $(r-1)^2(r+2)(r+3) = 0$, $r(x) = e^x$

43. $(r-1)^3(r+2)^2 r^4 = 0$, $r(x) = x + e^{-2x}$

44. $(r-1)^2(r+2)^3 r^5 = 0$, $r(x) = x + e^{-2x}$

45. $r^2(r-1)^2(r^2+4r+6) = 0$, $r(x) = x + e^x$

46. $r^3(r-1)(r^2+4r+6)^2 = 0$, $r(x) = x^2 + e^x$

47. $(r-1)(r+2)(r^2+1)^2 = 0$, $r(x) = \cos x + e^{-2x}$

48. $(r-1)^2(r+2)(r^2+1) = 0$, $r(x) = \sin x + e^{-2x}$

49. $(r-1)^3(r+2)^2(r^2+4) = 0$, $r(x) = \cos 2x + e^x$

50. $(r-1)^4(r+2)(r^2+4)^2 = 0$, $r(x) = \sin 2x + e^x$