What's Eigenanalysis?. Matrix eigenanalysis is a computational theory for the matrix equation $\mathbf{y} = A\mathbf{x}$. Here, we assume A is a 3×3 matrix.

The basis of eigenanalysis is **Fourier's Model**:

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \text{ implies} \\ \mathbf{y} &= A\mathbf{x} \\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3. \end{aligned} \tag{2}$$

The scale factors λ_1, λ_2, λ_3 and independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ depend only on A. Symbols c_1, c_2, c_3 stand for arbitrary numbers. This implies variable $\mathbf{x}$ exhausts all possible 3-vectors in R^3. Fourier's model is a replacement process:

$$A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

> To compute $A\mathbf{x}$ from $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, replace each vector $\mathbf{v}_i$ by its scaled version $\lambda_i\mathbf{v}_i$.

Fourier's model is said to **hold** provided there exist scale factors and independent vectors satisfying (2). Fourier's model is known to fail for certain matrices A.

Powers and Fourier's Model. Equation (2) applies to compute powers A^n of a matrix A using only the basic vector space toolkit. To illustrate, only the vector toolkit for R^3 is used in computing

$$A^5\mathbf{x} = x_1\lambda_1^5\mathbf{v}_1 + x_2\lambda_2^5\mathbf{v}_2 + x_3\lambda_3^5\mathbf{v}_3.$$

This calculation does not depend upon finding previous powers A^2, A^3, A^4 as would be the case by using matrix multiply.

Differential Equations and Fourier's Model. Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{array}{rcrcrcr} x_1' & = & x_1 & + & 3x_2, & & \\ x_2' & = & & & 2x_2 & - & x_3, \\ x_3' & = & & & & - & 5x_3. \end{array}$$

The general solution is given by the formula

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix},$$

which is related to Fourier's model by the symbolic formula

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3.$$

Fourier's model illustrated. Let

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = -5,$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Then Fourier's model holds (details later) and

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \quad \text{implies} \\ A\mathbf{x} &= c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \end{aligned}$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes $\mathbf{y} = A\mathbf{x}$ by scaling independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, which is a triad or **coordinate system**.

The subject of **eigenanalysis** discovers a coordinate system and scale factors such that Fourier's model holds. Fourier's model simplifies the matrix equation $\mathbf{y} = A\mathbf{x}$, through the formula

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

What's an Eigenvalue? It is a scale factor. An eigenvalue is also called a *proper value* or a *hidden value*. Symbols λ_1, λ_2, λ_3 used in Fourier's model are eigenvalues.

What's an Eigenvector? Symbols $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors*. They are assumed independent.

The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues).

A Key Example. Let $\mathbf{x}$ in R^3 be a data set variable with coordinates x_1, x_2, x_3 recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit $\mathbf{x}$-data into proper MKS units (meters-kilogram-second) $\mathbf{y}$-data via the equations

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= 0.001 x_2, \\ y_3 &= 0.01 x_3. \end{aligned} \tag{3}$$

Equations (3) are a **model** for changing units. Scaling factors $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 0.01$ are the **eigenvalues** of the model. To summarize:

> The **eigenvalues** of a model are **scale factors**. They are normally represented by symbols λ_1, λ_2, λ_3, ...

The data conversion problem (3) can be represented as $\mathbf{y} = A\mathbf{x}$, where the diagonal matrix A is given by

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = \frac{1}{1000},\ \lambda_3 = \frac{1}{100}.$$

Fourier's model for this matrix A is

$$A\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = c_1\lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\lambda_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$