An Illustration. The differential equation \( y'' - y = x + xe^x \) will be solved, verifying that \( y_h = c_1e^x + c_2e^{-x} \) and \( y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x \).

Solution:

Homogeneous solution. The characteristic equation \( r^2 - 1 = 0 \) has roots \( r = \pm 1 \). Recipe case 1 implies \( y_h = c_1e^x + c_2e^{-x} \).

Initial trial solution. The atoms of \( f(x) = x + xe^x \) are \( x \) and \( xe^x \). Differentiation of these atoms gives a new list of four distinct atoms \( 1, x, e^x, xe^x \). The undetermined coefficients will be assigned the symbols \( d_1, d_2, d_3, d_4 \). Then the initial trial solution is

\[
y = d_1 + d_2x + d_3e^x + d_4xe^x.
\]

Fixup rule. The atoms of \( y_h = c_1e^x + c_2e^{-x} \) form the list \( L = \{e^x, e^{-x}\} \). In \( y \), term \( d_3e^x \) contains atom \( e^x \) from list \( L \). Term \( d_3e^x \) and related atom term \( d_4xe^x \) are replaced by \( x(d_3e^x) \) and \( x(d_4xe^x) \), respectively. Terms \( d_1 \) and \( d_2x \) are unaffected by the fixup rule. The corrected trial solution is

\[
y = d_1 + d_2x + d_3xe^x + d_4x^2e^x.
\]

Because no term of \( y \) contains an atom in list \( L \), this is the final trial solution.

Substitute \( y \) into \( y'' - y = x + xe^x \). The details:
LHS = $y'' - y$

Left side of the equation.

$$= [y''_1 - y_1] + [y''_2 - y_2]$$

Let $y = y_1 + y_2$, $y_1 = d_1 + d_2 x$, $y_2 = d_3 xe^x + d_4 x^2 e^x$.

$$= [0 - y_1] + [2d_3 e^x + 2d_4 e^x + 4d_4 xe^x]$$

Use $y''_1 = 0$ and $y''_2 = y_2 + 2d_3 e^x + 2d_4 e^x + 4d_4 xe^x$.

$$= (-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x$$

Collect on distinct atoms.

Write out a $4 \times 4$ system. Because LHS = RHS and RHS = $x + xe^x$, the last display gives the relation

$$(-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x = x + xe^x. \quad (2)$$

Equate coefficients of matching atoms left and right to give the system of equations

$$
\begin{align*}
-d_1 & = 0, \\
-d_2 & = 1, \\
2d_3 + 2d_4 & = 0, \\
4d_4 & = 1.
\end{align*}
\quad (3)
$$

Atom matching effectively removes $x$ and changes the equation into a $4 \times 4$ linear system for symbols $d_1, d_2, d_3, d_4$. 

The technique is independence. To explain, independence of atoms means that a linear combination of atoms is uniquely represented, hence two such equal representations must have matching coefficients. Relation (2) says that two linear combinations of the same list of atoms are equal. Hence coefficients left and right in (2) must match, which gives $4 \times 4$ system (3).

Solve the equations. The $4 \times 4$ system must always have a unique solution. Equivalently, there are four lead variables and zero free variables. Solving by back-substitution gives $d_1 = 0$, $d_2 = -1$, $d_4 = 1/4$, $d_3 = -1/4$.

Report $y_p$. The trial solution with determined coefficients $d_1 = 0$, $d_2 = -1$, $d_3 = -1/4$, $d_4 = 1/4$ becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$ 

Report $y = y_h + y_p$. From above, $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$. Then $y = y_h + y_p$ is given by

$$y = c_1e^x + c_2e^{-x} - x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$ 

Answer check. Computer algebra system Maple is used.

```
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success is a report of zero.
```
Phase-amplitude conversion. Given a simple harmonic motion \( x(t) = c_1 \cos \omega t + c_2 \sin \omega t \), as in Figure 2, define amplitude \( A \) and phase angle \( \alpha \) by the formulas

\[
A = \sqrt{c_1^2 + c_2^2}, \quad c_1 = A \cos \alpha, \quad c_2 = A \sin \alpha.
\]

Then the simple harmonic motion has the phase-amplitude form

\[
x(t) = A \cos(\omega t - \alpha). \tag{4}
\]

To directly obtain (4) from trigonometry, use the trigonometric identity

\[
\cos(a - b) = \cos a \cos b + \sin a \sin b
\]

with \( a = \omega t \) and \( b = \alpha \). It is known from trigonometry that \( x(t) \) has period \( 2\pi/\omega \) and phase shift \( \alpha/\omega \). A full period is called a cycle and a half-period a semicycle. The frequency \( \omega/(2\pi) \) is the number of complete cycles per second, or the reciprocal of the period.
Figure 2. Simple harmonic oscillation $x(t) = A \cos(\omega t - \alpha)$, showing the period $2\pi/\omega$, the phase shift $\alpha/\omega$ and the amplitude $A$.

The phase shift is the amount of horizontal translation required to shift the cosine curve $\cos(\omega t - \alpha)$ so that its graph is atop $\cos(\omega t)$. To find the phase shift from $x(t)$, set the argument of the cosine term to zero, then solve for $t$.

To solve for $\alpha \geq 0$ in the equations $c_1 = A \cos \alpha$, $c_2 = A \sin \alpha$, first compute numerically by calculator the radian angle $\phi = \arctan(c_2/c_1)$, which is in the range $-\pi/2$ to $\pi/2$. Quadrantal angle rules must be applied when $c_1 = 0$, because calculators return an error code for division by zero. A common error is to set $\alpha$ equal to $\phi$. Not just the violation of $\alpha \geq 0$ results — the error is a fundamental one, due to trigonometric intricacies, causing us to consider the equations $c_1 = A \cos \alpha$, $c_2 = A \sin \alpha$ in order to construct the answer for $\alpha$:

$$\alpha = \begin{cases} 
\phi & (c_1, c_2) \text{ in quadrant I,} \\
\phi + \pi & (c_1, c_2) \text{ in quadrant II,} \\
\phi + \pi & (c_1, c_2) \text{ in quadrant III,} \\
\phi + 2\pi & (c_1, c_2) \text{ in quadrant IV.}
\end{cases}$$
**Cafe door.** Restaurant waiters and waitresses are accustomed to the cafe door, which partially blocks the view of onlookers, but allows rapid, collision-free trips to the kitchen – see Figure 3. The door is equipped with a spring which tries to restore the door to the equilibrium position $x = 0$, which is the plane of the door frame. There is a dampener attached, to keep the number of oscillations low.

![Cafe door diagram](image)

**Figure 3.** A cafe door on three hinges with dampener in the lower hinge. The equilibrium position is the plane of the door frame.

The top view of the door, Figure 4, shows how the angle $x(t)$ from equilibrium $x = 0$ is measured from different door positions.

![Door positions diagram](image)

**Figure 4.** Top view of a cafe door, showing the three possible door positions.
Pet door. Designed for dogs and cats, the small door in Figure 5 allows animals to enter and exit the house freely. Winter drafts and summer insects are the main reasons for pet doors. Owners argue that these doors decrease damage due to clawing and beating the door to get in and out. A pet door might have a weather seal and a security lock.

Figure 5. A pet door.
The equilibrium position is the plane of the door frame.

The pet door swings freely from hinges along the top edge. One hinge is spring–loaded with dampener. Like the cafe door, the spring restores the door to the equilibrium position while the dampener acts to eventually stop the oscillations. However, there is one fundamental difference: if the spring–dampener system is removed, then the door continues to oscillate! The cafe door model will not describe the pet door.
Cafe Door Model.

\[ Ix''(t) + cx'(t) + \kappa x(t) = 0. \] (5)

The removal of the spring \((\kappa = 0)\) causes the solution \(x(t)\) to be monotonic, which is a reasonable fit to a springless cafe door.
Pet Door Model.

Figure 7. A pet door.
The equilibrium position is the plane of the door frame.

For modeling purposes, the pet door can be compressed to a linearized swinging rod of length \( L \) (the door height). The torque \( I = mL^2/3 \) of the door assembly becomes important, as well as the linear restoring force \( kx \) of the spring and the viscous damping force \( cx' \) of the dampener. All considered, a suitable model is the pet door equation

\[
I x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right)x(t) = 0. \tag{6}
\]

Derivation of (6) is by equating to zero the algebraic sum of the forces. Removing the dampener and spring \((c = k = 0)\) gives a harmonic oscillator \( x''(t) + \omega^2 x(t) = 0 \) with \( \omega^2 = 0.5mgL/I \), which establishes sanity for the modeling effort. Equation (6) is formally the cafe door equation with an added linearization term \( 0.5mgLx(t) \) obtained from \( 0.5mgL \sin x(t) \).
Classifying Damped Models. Consider a differential equation

\[ ay'' + by' + cy = 0 \]

with constant coefficients \( a, b, c \). It has characteristic equation \( ar^2 + br + c = 0 \) with roots \( r_1, r_2 \).

<table>
<thead>
<tr>
<th>Classification</th>
<th>Defining properties</th>
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| Overdamped         | Distinct real roots \( r_1 \neq r_2 \)  
                     | Positive discriminant 
                     | \( x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \) 
                     | = exponential \( \times \) monotonic function |
| Critically damped  | Double real root \( r_1 = r_2 \)  
                     | Zero discriminant  
                     | \( x = c_1 e^{r_1 t} + c_2 t e^{r_1 t} \) 
                     | = exponential \( \times \) monotonic function |
| Underdamped        | Complex conjugate roots \( \alpha \pm i \beta \)  
                     | Negative discriminant  
                     | \( x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \) 
                     | = exponential \( \times \) harmonic oscillation |
Figure 8. Pure resonance.

Graphed are the envelope curves $x = \pm t$ and the solution $x(t) = t \sin 4t$ of the equation $x''(t) + 16x(t) = 8 \cos \omega t$, where $\omega = 4$.

The notion of pure resonance in the differential equation

$$x''(t) + \omega^2_0 x(t) = F_0 \cos(\omega t) \quad (7)$$

is the existence of a solution that is unbounded as $t \to \infty$. We already know that for $\omega \neq \omega_0$, the general solution of (7) is the sum of two harmonic oscillations, hence it is bounded. Equation (7) for $\omega = \omega_0$ has by the method of undetermined coefficients the unbounded oscillatory solution $x(t) = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$. To summarize:

Pure resonance occurs exactly when the natural internal frequency $\omega_0$ matches the natural external frequency $\omega$, in which case all solutions of the differential equation are unbounded.

In Figure 8, this is illustrated for $x''(t) + 16x(t) = 8 \cos 4t$, which in (7) corresponds to $\omega = \omega_0 = 4$ and $F_0 = 8$. 
**Real-World Damping Effects.** The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching

\[ \omega = \omega_0 \equiv \sqrt{k/m} \]

characterizes the event. This ideal situation never happens in the physical world, because *damping is always present*. In the presence of damping \( c > 0 \), it can be established that *only bounded solutions exist* for the forced spring-mass system

\[ mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t. \]  

(8)

Our intuition about resonance seems to vaporize in the presence of damping effects. But not completely. Most would agree that the undamped intuition is correct when the damping effects are nearly zero.

**Practical resonance** is said to occur when the external frequency \( \omega \) has been tuned to produce the largest possible solution. It can be shown that this happens for the condition

\[ \omega = \sqrt{k/m - c^2/(2m^2)}, \quad k/m - c^2/(2m^2) > 0. \]

(9)

**Pure resonance** \( \omega = \omega_0 \equiv \sqrt{k/m} \) is the limiting case obtained by setting the damping constant \( c \) to zero in condition (9). This strange but predictable interaction exists between the damping constant \( c \) and the size of solutions, relative to the external frequency \( \omega \), even though all solutions remain bounded.