Atom List. L. Euler supplies us with a basic result, which tells us how to find the list of distinct atoms.

Theorem 19 (Euler)
The function $e^{rx}$ is a solution of a linear constant coefficient differential equation if and only if $r$ is a root of the characteristic equation.

More generally, the list of distinct atoms $e^{rx}$, $xe^{rx}$, \ldots, $x^ke^{rx}$ consists of solutions if and only if $r$ is a root of the characteristic equation of multiplicity $k + 1$.

If $r = \alpha + i\beta$ is a complex root of multiplicity $k + 1$, then the formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies

$$e^{rx} = e^{\alpha x} \cos (\beta x) + ie^{\alpha x} \sin (\beta x).$$

Therefore, the $2k + 2$ distinct atoms listed below are independent solutions of the differential equation:

$$e^{\alpha x} \cos (\beta x), \quad xe^{\alpha x} \cos (\beta x), \quad \ldots, \quad x^ke^{\alpha x} \cos (\beta x),$$

$$e^{\alpha x} \sin (\beta x), \quad xe^{\alpha x} \sin (\beta x), \quad \ldots, \quad x^ke^{\alpha x} \sin (\beta x)$$
Example (First Order) Solve $2y' + 5y = 0$ by using the $n$th order recipe, showing $y_h = c_1 e^{-5x/2}$.

Solution: The characteristic equation is $2r + 5 = 0$ with real root $r = -5/2$ and corresponding atom $e^{rx}$ given explicitly by $e^{-5x/2}$. Euler’s Theorem was applied here. The order of the differential equation is 1, so we have found all atoms. The general solution $y_h$ is written by multiplying the atom list by constants $c_1, c_2, \ldots$, and therefore $y_h = c_1 e^{-5x/2}$. 


2 Example (Second Order I) Solve \( y'' + 2y' + y = 0 \) by using the \( n \)th order recipe, showing \( y_h = c_1 e^{-x} + c_2 x e^{-x} \).

**Solution:** The characteristic equation is \( r^2 + 2r + 1 = 0 \) with double real root \( r = -1, -1 \). Euler’s Theorem applies to report atom list \( e^{rx}, xe^{rx} \), given explicitly by \( e^{-x}, xe^{-x} \). The order of the differential equation is 2, so we have found all atoms. The general solution \( y_h \) is written by multiplying the atom list by constants \( c_1, c_2, \ldots \), and therefore \( y_h = c_1 e^{-x} + c_2 x e^{-x} \).
3 Example (Second Order II) Solve $y'' + 3y' + 2y = 0$ by using the $n$th order recipe, showing $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

**Solution:** The characteristic equation is $r^2 + 3r + 2 = 0$ with distinct real roots $r_1 = -1$, $r_2 = -2$. Euler’s Theorem applies to report atom list $e^{r_1 x}$, $e^{r_2 x}$, given explicitly by $e^{-x}$, $e^{-2x}$. The order of the differential equation is 2, so we have found all atoms. The general solution $y_h$ is written by multiplying the atom list by constants $c_1$, $c_2$, ..., and therefore $y_h = c_1 e^{-x} + c_2 e^{-2x}$. 
4 Example (Second Order III) Solve \(y'' + 2y' + 5y = 0\) by using the \(n\)th order recipe, showing \(y_h = c_1e^{-x}\cos 2x + c_2xe^{-x}\sin 2x\).

Solution: The characteristic equation is \(r^2 + 2r + 5 = 0\) with complex conjugate roots \(r_1 = -1 + 2i, r_2 = -1 - 2i\). Euler’s Theorem applies to report an atom list \(e^{\alpha x}\cos \beta x, e^{\alpha x}\sin \beta x\), where \(\alpha = -1, \beta = 2\) are the real and imaginary parts of the root \(\alpha + i\beta = -1 + 2i\) (then \(\alpha = -1, \beta = 2\)). The atom list is given explicitly by \(e^{-x}\cos 2x, e^{-x}\sin 2x\). The order of the differential equation is 2, so we have found all atoms. The lesson: applying Euler’s theorem to the second conjugate root \(-1 - 2i\) will produce no new atoms. The general solution \(y_h\) is written by multiplying the atom list by constants \(c_1, c_2, \ldots\), and therefore \(y_h = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x\).
5 Example (Third Order I) Solve $y''' - y' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 e^x + c_3 e^{-x}$.

Solution: The characteristic equation is $r^3 - r = 0$ with real roots $r_1 = 0$, $r_2 = 1$, $r_3 = -1$. Euler’s Theorem applies to report atom list $e^{r_1 x}$, $e^{r_2 x}$, $e^{r_3 x}$ given explicitly by $e^{0x}$, $e^x$, $e^{-x}$. The order of the differential equation is 3, so we have found all atoms. The general solution $y_h$ is written by multiplying the atom list by constants $c_1$, $c_2$, ..., and therefore $y_h = c_1 e^{0x} + c_2 e^x + c_3 e^{-x}$. Convention dictates replacing $e^{0x}$ by 1 in the final equation.
6 Example (Third Order II) Solve $y''' - y'' = 0$ by using the $n$th order recipe, showing $y_h = c_1 + c_2x + c_3e^x$.

Solution: The characteristic equation is $r^3 - r^2 = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. Euler’s Theorem applies to report atom list $e^{r_1x}$, $xe^{r_1x}$, $e^{r_3x}$ given explicitly by $e^{0x}$, $xe^{0x}$, $e^x$. The order of the differential equation is 3, so we have found all atoms. The general solution $y_h$ is written by multiplying the atom list by constants $c_1$, $c_2$, . . . , and therefore $y_h = c_1e^{0x} + c_2xe^{0x} + c_3e^x$. Convention dictates replacing $e^{0x}$ by 1 in the final equation.
7 Example (Fourth Order) Solve $y^{iv} - y'' = 0$ by using the $n$th order recipe, showing $y_h = c_1 + c_2x + c_3e^x + c_4e^{-x}$.

**Solution:** The characteristic equation is $r^4 - r^2 = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$, $r_4 = -1$. Euler’s Theorem applies to obtain the atom list $e^{r_1x}$, $xe^{r_1x}$, $e^{r_3x}$, $e^{r_4x}$, given explicitly by $e^{0x}$, $xe^{0x}$, $e^x$, $e^{-x}$. The order of the differential equation is 4, so we have found all atoms. The general solution $y_h$ is written by multiplying the atom list by constants $c_1$, $c_2$, ..., and therefore $y_h = c_1e^{0x} + c_2xe^{0x} + c_3e^x + c_4e^{-x}$. Convention replaces $e^{0x}$ by 1 in the final equation.