Atom List. L. Euler supplies us with a basic result, which tells us how to find the list of distinct atoms.

Theorem 19 (Euler)

The function e^{rx} is a solution of a linear constant coefficient differential equation if and only if r is a root of the characteristic equation.

More generally, the list of distinct atoms e^{rx}, xe^{rx}, ..., $x^k e^{rx}$ consists of solutions if and only if r is a root of the characteristic equation of multiplicity $k+1$.

If $r = \alpha + i\beta$ is a complex root of multiplicity $k+1$, then the formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$e^{rx} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x).$$

Therefore, the $2k+2$ distinct atoms listed below are independent solutions of the differential equation:

$$\begin{array}{llll} e^{\alpha x}\cos(\beta x), & xe^{\alpha x}\cos(\beta x), & \ldots, & x^k e^{\alpha x}\cos(\beta x), \\ e^{\alpha x}\sin(\beta x), & xe^{\alpha x}\sin(\beta x), & \ldots, & x^k e^{\alpha x}\sin(\beta x) \end{array}$$

1 Example (First Order) Solve $2y' + 5y = 0$ by using the nth order recipe, showing $y_h = c_1 e^{-5x/2}$.

Solution: The characteristic equation is $2r + 5 = 0$ with real root $r = -5/2$ and corresponding atom e^{rx} given explicitly by $e^{-5x/2}$. Euler's Theorem was applied here. The order of the differential equation is 1, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1 e^{-5x/2}$.

2 Example (Second Order I) Solve $y'' + 2y' + y = 0$ by using the nth order recipe, showing $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

Solution: The characteristic equation is $r^2 + 2r + 1 = 0$ with double real root $r = -1, -1$. Euler's Theorem applies to report atom list e^{rx}, xe^{rx}, given explicitly by e^{-x}, xe^{-x}. The order of the differential equation is 2, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

3 Example (Second Order II) Solve $y''+3y'+2y=0$ by using the nth order recipe, showing $y_h = c_1e^{-x} + c_2e^{-2x}$.

Solution: The characteristic equation is $r^2+3r+2 = 0$ with distinct real roots $r_1 = -1$, $r_2 = -2$. Euler's Theorem applies to report atom list e^{r_1x}, e^{r_2x}, given explicitly by e^{-x}, e^{-2x}. The order of the differential equation is 2, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, . . . , and therefore $y_h = c_1e^{-x} + c_2e^{-2x}$.

4 Example (Second Order III) Solve $y''+2y'+5y=0$ by using the nth order recipe, showing $y_h = c_1e^{-x}\cos 2x + c_2xe^{-x}\sin 2x$.

Solution: The characteristic equation is $r^2+2r+5=0$ with complex conjugate roots $r_1 = -1+2i$, $r_2 = -1-2i$. Euler's Theorem applies to report an atom list $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$, where $\alpha=-1$, $\beta=2$ are the real and imaginary parts of the root $\alpha+i\beta=-1+2i$ (then $\alpha=-1$, $\beta=2$). The atom list is given explicitly by $e^{-x}\cos 2x$, $e^{-x}\sin 2x$. The order of the differential equation is 2, so we have found all atoms. The lesson: applying Euler's theorem to the second conjugate root $-1-2i$ will produce no new atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x$.

5 Example (Third Order I) Solve $y''' - y' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 e^x + c_3 e^{-x}$.

Solution: The characteristic equation is $r^3 - r = 0$ with real roots $r_1 = 0$, $r_2 = 1$, $r_3 = -1$. Euler's Theorem applies to report atom list $e^{r_1 x}$, $e^{r_2 x}$, $e^{r_3 x}$ given explicitly by e^{0x}, e^x, e^{-x}. The order of the differential equation is 3, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1 e^{0x} + c_2 e^x + c_3 e^{-x}$. Convention dictates replacing e^{0x} by 1 in the final equation.

6 Example (Third Order II) Solve $y''' - y'' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 x + c_3 e^x$.

Solution: The characteristic equation is $r^3 - r^2 = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. Euler's Theorem applies to report atom list $e^{r_1 x}$, $xe^{r_1 x}$, $e^{r_3 x}$ given explicitly by e^{0x}, xe^{0x}, e^x. The order of the differential equation is 3, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^x$. Convention dictates replacing e^{0x} by 1 in the final equation.

7 Example (Fourth Order) Solve $y^{iv} - y'' = 0$ by using the nth order recipe, showing $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Solution: The characteristic equation is $r^4 - r^2 = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$, $r_4 = -1$. Euler's Theorem applies to obtain the atom list $e^{r_1 x}$, $xe^{r_1 x}$, $e^{r_3 x}$, $e^{r_4 x}$, given explicitly by e^{0x}, xe^{0x}, e^x, e^{-x}. The order of the differential equation is 4, so we have found all atoms. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, ..., and therefore $y_h = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^x + c_4 e^{-x}$. Convention replaces e^{0x} by 1 in the final equation.