

Atoms. An **atom** is a term of one of the following types:

$$x^n, x^n e^{ax}, x^n e^{ax} \cos bx, x^n e^{ax} \sin bx.$$

The symbol n is an integer $0, 1, 2, \dots$ and a, b are real numbers with $b > 0$. In particular,

$$1, x, x^2, \dots, x^k \text{ are atoms.}$$

The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom e^x . Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms x and x^2 are indeed atoms. Terms like e^{x^2} , $\ln|x|$ and $x/(1+x^2)$ are not atoms, nor are they constructed from atoms.

The **recipe** justifies the following result for the special case of second order differential equations, because e^{r_1x} , e^{r_2x} , xe^{r_1x} , $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are atoms.

Theorem 13 (Homogeneous Solution y_h and Atoms)

Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of atoms.

Theorem 14 (Particular Solution y_p and Atoms)

A linear non-homogeneous differential equation with constant coefficients having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 15 (General Solution y and Atoms)

A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{a linear combination of atoms.}$$

While the third theorem easily follows from the first two, the first and second theorem have yet to be verified for all cases. The first will follow directly from the higher order recipe, while the second follows from the method of undetermined coefficients, *infra*.

Independence. Linear algebra defines a list of functions f_1, \dots, f_k to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \dots + c_k f_k(x) \quad \text{for all } x$$

implies $c_1 = c_2 = \dots = c_k = 0$.

Geometrically, list f_1, f_2 is independent provided both are not the graph $y = 0$ and it is impossible to deform one graph into the other by re-scaling of the y -axis. For instance, the line $y = 1$ cannot be deformed into the 45° line $y = x$ by re-scaling.

Theorem 16 (Independence and Atoms)

A list of finitely many distinct atoms is linearly independent. In particular, the list of distinct atoms

$$1, x, x^2, \dots, x^k$$

is linearly independent.

How to solve n -th order equations. Picard's existence–uniqueness theorem says that $y''' + 2y'' + y = 0$ has general solution y constructed from 3 solutions of this differential equation. More precisely, the general solution of an n -th order linear differential equation is constructed from n solutions of the equation.

Linear algebra says that the dimension of the solution set is this same fixed number n . Once n independent solutions are found for the differential equation, the search for the general solution has ended: y must be a linear combination of these n independent solutions.

Because of the preceding structure theorems, we have reduced our search for the general solution as follows:

Find n distinct atoms which are solutions of the differential equation.

Euler supplies us with a basic result, which tells us how to find the list of distinct atoms.

Theorem 17 (Euler)

The function e^{rx} is a solution of a linear constant-coefficient differential equation if and only if r is a root of the characteristic equation.

More generally, the $k + 1$ distinct atoms

$$e^{rx}, xe^{rx}, \dots, x^k e^{rx}$$

are solutions if and only if r is a root of the characteristic equation of multiplicity $k + 1$.

Theorem 18 (Complex Roots)

If $r = \alpha + i\beta$ is a complex root of multiplicity $k + 1$, then the formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies

$$e^{rx} = e^{\alpha x} \cos(\beta x) + ie^{\alpha x} \sin(\beta x)$$

and therefore the $2k + 2$ distinct atoms

$$\begin{aligned} e^{\alpha x} \cos(\beta x), & \quad xe^{\alpha x} \cos(\beta x), \quad \dots, \quad x^k e^{\alpha x} \cos(\beta x), \\ e^{\alpha x} \sin(\beta x), & \quad xe^{\alpha x} \sin(\beta x), \quad \dots, \quad x^k e^{\alpha x} \sin(\beta x) \end{aligned}$$

are solutions of the differential equation.