1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let S be the vector space of all continuous functions defined on  $(-\infty, \infty)$ . Define V to be the set of all functions f(x) in S such that  $\int_0^1 f(x)xe^{-x}dx = 0$ . Prove that V is a subspace of S.

(b) [30%] Find a  $4 \times 5$  augmented matrix representing four equations for the constants a, b, c, d in the partial fractions decomposition for the fraction given below. To save time, **do not solve for** a, b, c, d!

$$\frac{2x-1}{(x-2)^2(x^2-4x+8)}$$

(b) [70%] Solve for the unknowns a, b, c, d in the system of equations below by augmented matrix RREF methods, showing all details.

**Solution 1(a).** Use the subspace criterion: (a) Given f and g in V, write details to show f+g is in V; (b) Given f in V and k constant, write details to show kf is in V. Let  $h(x) = xe^{-x}$ , which is a function in S. Details for (a): Given  $\int_0^1 f(x)h9x)dx = 0$  and  $\int_0^1 g(x)h(x)dx = 0$ , add the equations to obtain the equation  $\int_0^1 (f(x)+g(x))h(x)dx = 0$ . This finishes (a). Details for (b): Given  $\int_0^1 f(x)h(x)dx = 0$  and k constant, multiply the equation by k and re-arrange factors to obtain the new equation  $\int_0^1 (kf(x))h(x)dx = 0$ . This proves (b).

Solution 1(b). The decomposition might be

$$\frac{2x-1}{(x-2)^2(x^2-4x+8)} = \frac{a}{x-2} + \frac{b}{(x-2)^2} + \frac{c(x-2)+2d}{x^2-4x+8}$$

although there are other possibilities. Clear the fractions. Set x = 2 to get one equation for the constants. Choose 3 other values for x to obtain three other equations. Display the system of equations.

Solution 1(c). The answer is 
$$\begin{pmatrix} 2+4t_-2t_2 \\ -2t_1 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$



- 2. (ch5) Complete all parts.
  - (a) [30%] Given 4x''(t) + 2x'(t) + x(t) = 0, which represents a damped spring-mass system with m = 4, c = 2, k = 1, **solve** the differential equation [20%] and **classify** the answer as over-damped. critically damped or under-damped [10%].
  - (b) [70%] Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation  $x'' + 2x' + x = 5\cos(2t)$ .

(a) 
$$4r^2 + 2r + 1 = 0$$
 char eq  
has roots  $-\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$   
(10%) undn-damped  
[20%] Sol  $y = c, e^{-t/4} \cos(\sqrt{3}t/4) + c_2 e^{-t/4} \sin(\sqrt{3}t/4)$ 

(b) 
$$x = d_1 \cos 2t + d_2 \sin 2t$$
  
 $= \text{ corrected trial } 50|$   
 $-4d_1 \cos 2t - 4d_2 \sin 2t + 2(-2d_1 \text{ Am } 2t + 2d_2 \cos 2t) + d_1 \cos 2t + d_2 \text{ Am } 2t}$   
 $= 5 \cos(2t)$   
 $\int -4d_1 + 4d_2 + d_1 = 5$   
 $-4d_2 - 4d_1 + d_2 = 0$   $\Leftrightarrow$   $\begin{pmatrix} -3 & 4 & 4 & 4d_2 & 4d_$ 

$$\chi = d, \cos 2t + d_2 \sin 2t$$

$$\chi_{p} = -\frac{3}{5} \cos 2t + \frac{4}{5} \sin 2t$$

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- 3. (ch5) Complete all parts below.
  - (a) [75%] Determine for  $y^v 4y''' = xe^{2x} + x^2(1+x) + 2\cos x$  the **corrected** trial solution for  $y_p$  according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps  $\boxed{1}$  and  $\boxed{2}$ , but skip steps  $\boxed{3}$  and  $\boxed{4}$ )! Undocumented details will be considered guessing, which earns no credit.
  - (b) [10%] Using the recipe for second order constant-coefficient differential equations, write out a basis for the solution space of the equation y'' + 4y' y = 0.
  - (c) [15%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is  $(r-1)^2(r^2-1)^2(r^2+4)^2=0$ .

(b) 
$$r^{2}_{+} + r - 1 = 0$$
 $r = -\frac{4}{2} \pm \frac{1}{2} \sqrt{16} + 4$ 
 $= -2 \pm \sqrt{5}$ 
Basis =  $e^{-2t + \sqrt{5}t}$ ,  $e^{-2t - \sqrt{5}t}$ ?

$$O(r-1)^{4}(r+1)^{2}(r^{2}+4)^{2}=0$$

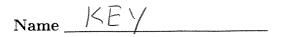
$$W_{1} = C_{1} + C_{1} \times + C_{3} \times^{2} + C_{4} \times^{3}$$

$$W_{2} = C_{7} + C_{8} \times$$

$$W_{3} = C_{7} + C_{8} \times$$

$$W_{4} = C_{9} + C_{9} \times$$

Use this page to start your solution. Staple extra pages as needed.



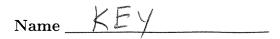
- 4. (ch6) Complete all of the items below.
  - (a) [30%] Find the eigenvalues of the matrix  $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ . To save time. **do not** find eigenvectors!
  - (b) [70%] Given  $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ , then there exists an invertible matrix P and a diagonal matrix Dsuch that AP = PD. Find one possible column of P.

Eigenvalue = 
$$1, 1, 1\pm 3i$$

(a) 
$$\lambda = 1$$
 is an eigenvalue. Fixed an eigenpair  $(1, \vec{v})$ .

$$\begin{pmatrix} 0 & 1 & -1 & | & 0 \\ 0 & 4 & 1 & | & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & | & -1 & | & 0 \\ 0 & 0 & -5 & | & 0 \\ 0 & 0 & 5 & | & 0 \end{pmatrix}$$

$$\frac{7}{3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
also  $\begin{pmatrix} 4 \\ -\frac{1}{3} \end{pmatrix}$ 
and  $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ 



5. (ch6) Complete all parts below.

Consider a given  $3 \times 3$  matrix A having three eigenpairs

$$6, \left(\begin{array}{c}1\\2\\0\end{array}\right); \quad 4, \left(\begin{array}{c}-1\\1\\1\end{array}\right); \quad 1, \left(\begin{array}{c}1\\-2\\0\end{array}\right).$$

- (a) [50%] Display the general solution  $\mathbf{x}(t)$  of the system  $\mathbf{x}' = A\mathbf{x}$  in vector form.
- (b) [20%] Write a matrix algebra formula for the matrix A of (a) above. To save time, do not evaluate anything.
- (c) [30%] Describe precisely Fourier's simplification method for the equation  $\mathbf{y} = A\mathbf{x}$ , using the matrix A of (a) above.

(a) 
$$\vec{\chi}(t) = c_1(\frac{1}{3})e^{6t} + c_2(\frac{-1}{1})e^{4t} + c_3(\frac{-1}{3})e^{4t}$$

Use this page to start your solution. Staple extra pages as needed.