Name. KEY

Exam 2 2250-2 10:45

1. (rref)

Determine a, b such that (1) the system has no solution and (2) the system has infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ 2x & + & 10y & + & 8z & = & 3 \\ 3x & + & ay & + & 3bz & = & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 10 & 8 & 3 \\ 3 & a & 3b & 2 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & a-6 & 3b-3 & -1 \end{array}\right) \quad \begin{array}{l} \text{combo}(1,2,-2) \\ \text{combo}(1,3,-3) \end{array}$$

$$\cong \left(\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1/6 \\ 0 & a-6 & 3b-3 & -1 \end{array}\right) \quad \text{mult}(2, 1/6)$$

$$\cong \left(\begin{array}{cccc} 1 & 0 & -1 & 2/3 \\ 0 & 1 & 1 & 1/6 \\ 0 & 0 & \begin{array}{c}3b-3 \\ +6-a\end{array} & -1+\frac{6-a}{6} \end{array}\right) \quad \begin{array}{l} \text{combo}(2,1,-2) \\ \text{combo}(2,3,6-a) \end{array}$$

$$\cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2/3 \\ 0 & 1 & 1 & 1/6 \\ 0 & 0 & x & y \end{array}\right) \quad \begin{array}{l} x = 3b+3-a \\ y = -a/6 \end{array}$$

If $x \neq 0$, then three lead vars exist $\Rightarrow$ unique sol

If $x = 0$ but $y \neq 0$, then no sol because of signal equation

If $x = 0$ and $y = 0$, then one free var $\Rightarrow$ ∞-many sols.

answer

(1) No solution for $3b+3-a=0$ and $-a/6 \neq 0$

(2) ∞-many sols for $3b+3-a=0$ and $-a/6=0$

This is the top page. Staple extra pages to this page.

Name. KEY **Exam 2 2250-2 10:45**

2. **(vector spaces)** Do **two** of the following but not three.

(a) [50%] Let V be the vector space of functions $f(t) = c_1 + c_2e^{2t} + c_3e^{4t} + 3c_4$, for all values of c_1, c_2, c_3, c_4. Report a basis for V.

(b) [50%] Prove by means of the subspace criterion (Theorem 1, Edwards-Penney) that the set S of all fixed vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ with $v_1 + v_2 = 0$ is a subspace.

(c) [50%] Find a basis of 3-vectors for the solution space of the system of equations

$$\begin{aligned} x + 3y - 3z &= 0, \\ y + 2z &= 0, \\ x + 4y - z &= 0, \end{aligned}$$

(a) Basis is contained in $\{\partial_{c_1} f, \partial_{c_2} f, \partial_{c_3} f, \partial_{c_4} f\} = \{1, e^{2t}, e^{4t}, 3\}$

Since 3 depends on the others, the largest independent set is $\boxed{\{1, e^{2t}, e^{4t}\} = \text{Basis}}$

(b) write the homogeneous restriction equation as $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or $A\vec{v} = \vec{0}$ with $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Apply subspace criterion:

(1) $\vec{v}_1, \vec{v}_2$ in $S \Rightarrow A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{0}$

$\Rightarrow A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$

$\Rightarrow \vec{v}_1 + \vec{v}_2$ in S.

(2) $\vec{v}$ in S and k = constant $\Rightarrow A\vec{v} = \vec{0}$, k = const

$\Rightarrow A(k\vec{v}) = k(A\vec{v}) = k(\vec{0}) = \vec{0}$

$\Rightarrow k\vec{v}$ in S.

(c) $\left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 4 & -1 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -9 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x = 9t_1 \\ y = -2t_1 \\ z = t_1 \end{cases} \Rightarrow \boxed{\text{Basis} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}}$

This is the top page. Staple extra pages to this page.

Name. KEY

3. **(independence)** Do either (a) or (b) but not both.

(a) [100%] Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 5 \\ -5 \\ 0 \\ -5 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 6 \\ -3 \\ 0 \\ -2 \end{pmatrix}.$$

(b) [100%] Let matrix D be given and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors with $D\mathbf{a}$, $D\mathbf{b}$, $D\mathbf{c}$, $D\mathbf{d}$ independent. Prove that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are independent. Don't do this problem if you did (a)!

(a) $$A = \begin{pmatrix} 1 & -2 & 5 & 4 & 6 \\ -1 & 2 & -5 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -5 & 0 & -2 \end{pmatrix} \cong \begin{pmatrix} 1 & -2 & 5 & 4 & 6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ combo(1,2,1) Combo(1,4,1) swap(3,4)

$$\cong \begin{pmatrix} 1 & -2 & 5 & 4 & 6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ mult + combo

$\cong$ rref(A) with leading ones in Cols 1, 4.

$\boxed{\vec{a} \text{ and } \vec{d} \text{ are largest indep set}}$

(b) Let c_1, c_2, c_3, c_4 satisfy the equation

$$c_1\vec{a} + c_2\vec{b} + c_3\vec{c} + c_4\vec{d} = \vec{0}$$

Multiply by D to give

$$c_1 D\vec{a} + c_2 D\vec{b} + c_3 D\vec{c} + c_4 D\vec{d} = \vec{0}$$

By independence of $D\vec{a}, D\vec{b}, D\vec{c}, D\vec{d}$ we have

$$c_1 = c_2 = c_3 = c_4 = 0.$$

The proof is complete.

This is the top page. Staple extra pages to this page.

Name. KEY

Exam 2 2250-2 10:45

4. **(determinants and elementary matrices)**
Assume given two invertible 4×4 matrices A, B. Let elementary matrices E_1, E_2, E_3 be given, with E_1 a swap, E_2 a combination and E_3 a multiply, with multiplier $1/16$, and assume $B = E_1^{-1}E_2E_3A$. Explain precisely why $\det(4BA^{-1}) = -16$.

$$BA^{-1} = E_1^{-1}E_2E_3$$

$$4BA^{-1} = (4I)E_1E_2E_3 \qquad \text{because } E_1^{-1} = E_1 \text{ for swaps}$$

$$\det(4BA^{-1}) = \det(4I)\det(E_1)\det(E_2)\det(E_3) \qquad \text{prod rule}$$

$$= \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} (-1)(1)\left(\frac{1}{16}\right) \qquad \text{because } \begin{cases} \det E_1 = -1 \\ \det E_2 = 1 \\ \det E_3 = 1/16 \end{cases}$$

$$= (16)(16)(-1)\frac{1}{16}$$

$$= -16$$

This is the top page. Staple extra pages to this page.

Name. KEY **Exam 2 2250-2 10:45**

5. (inverses and Cramer's rule)

(a) [75%] Write a determinant formula for x_3 in $A\mathbf{u} = \mathbf{b}$ according to Cramer's rule, but don't find the value of the determinants. Use matrix A, column vectors $\mathbf{u}$ and $\mathbf{b}$ given below.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 2 & -2 & 1 & 0 \\ 3 & 1 & 7 & 2 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}$$

(b) [25%] The four determinant rules for computing the value of any determinant are called *triangular, swap, combo, mult.* State one determinant rule which can be proved from the *mult* rule. Don't give a proof.

(a) $x_3 = \dfrac{\Delta_3}{\Delta}$ $\quad A = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 2 & -2 & 1 & 0 \\ 3 & 1 & 7 & 2 \end{vmatrix}$ $\quad \Delta_3 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 2 & 5 \\ 2 & -2 & 4 & 0 \\ 3 & 1 & -1 & 2 \end{vmatrix}$

(b) If A has a row of zeros, then $\det(A) = 0$.

This is the top page. Staple extra pages to this page.