Name. KEY

Applied Differential Equations 2250-2
Midterm Exam 1
Wednesday, 28 September 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = xe^{-x} + \csc^2 x + \cot^2 x + \dfrac{16x^4}{1+4x^2}$.

$$y = \int (RHS)\,dx$$

$$= c + \int xe^{-x}\,dx + \int (2\csc^2 x - 1)\,dx + \int \left\{ 4x^2 - 1 + \frac{1}{1+4x^2} \right\} dx$$

$\sin^2 x + \cos^2 x = 1$
$1 + \cot^2 x = \csc^2 x$

$$\begin{array}{r} 4x^2 - 1 \\ 1+4x^2 \overline{) 16x^4} \\ \underline{16x^4 + 4x^2} \\ -4x^2 \\ \underline{-4x^2 - 1} \\ 1 \end{array}$$

$$= c + uv - \int v\,du + (-2\cot x) - x + \frac{4}{3}x^3 - x + \int \frac{dx}{1+4x^2}$$

$u = x \quad du = dx$
$dv = e^{-x}dx \quad v = -e^{-x}$

$w = 2x$
$dw = 2dx$

$$= c + (-x)e^{-x} - \int -e^{-x}dx - 2\cot x - x + \frac{4}{3}x^3 - x + \int \frac{dw/2}{1+w^2}$$

$$= \boxed{c - xe^{-x} - e^{-x} - 2\cot x - 2x + \frac{4}{3}x^3 + \frac{1}{2}\tan^{-1}(2x)}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY 2250-2

2. **(Separable Equation Test)**
The problem

$$y' = x\left(e^{2x}x^{2/3}ye^{y} + \sin(x)ye^{y}\right) - x\sin(x)\sin(y) - e^{2x}x^{5/3}\sin(y)$$

may or may not be separable. If it is, then write formulae for F, G which decompose the problem as $y' = F(x)G(y)$. Otherwise, explain in detail why it fails to be separable. Do not solve for y!

$F(x) = \dfrac{f(x,y_0)}{f(x_0,y_0)}$ $\qquad G(y) = f(x_0,y)$

If $x_0 = y_0 = \frac{\pi}{2}$, Then $f(x_0,y_0) = \frac{\pi}{2}\left(e^{\pi}\frac{\pi}{2}^{2/3}\frac{\pi}{2}e^{\pi/2} + \frac{\pi}{2}e^{\pi/2}\right) - \frac{\pi}{2} - e^{\pi}\frac{\pi}{2}^{5/3} \neq 0$

$G(y) = x_0^{2/3+1}e^{2x_0}ye^{y} + x_0\sin x_0\, ye^{y} - \left(x_0\sin x_0 + e^{2x_0}x_0^{5/3}\right)\sin(y)$

$= c\,(ye^{y} - \sin y)$, where $c = x_0\sin x_0 + e^{2x_0}x_0^{5/3}$

Choose
$F(x) = xe^{2x}x^{2/3} + x\sin x$, $\quad G(y) = ye^{y} - \sin y$
Then

$F(x)G(y) = (x^{5/3}e^{2x} + x\sin x)(ye^{y} - \sin y)$

$= x^{5/3}e^{2x}ye^{y} + xy\sin y\, e^{y} - x^{5/3}e^{2x}\sin y - x\sin x\sin y$

$= f(x,y)$

The eq is <u>separable</u>.

- Second solution: choose $x_0 = \pi$, $y_0 = 2\pi$. Do the standard F, G formulae. Show $FG = f$.
- Third solution: Choose $x_0 = y_0 = \pi$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

2250-2

3. (Solve a Separable Equation)
Given $yy' = \dfrac{6x^3+12x}{1+x}(1-4y^2)$,
(a) Find all equilibrium solutions,
(b) Find the non-equilibrium solution in implicit form.
Do not solve for y explicitly.

$$F(x) = \frac{6x^3+12x}{1+x} = 6x^2-6x+18 + \frac{-18}{1+x}$$

$$G(y) = \frac{1-4y^2}{y}$$

$$y' = F(x)G(y)$$

Division algorithm

$$\begin{array}{r} 6x^2-6x+18 \\ 1+x \overline{)\, 6x^3+12x} \\ \underline{6x^3+6x^2} \\ -6x^2+12x \\ \underline{-6x^2-6x} \\ 18x \\ \underline{18x+18} \\ -18 \end{array}$$

(a) [15] • Equil sols satisfy $G(y)=0$.

$$\boxed{y = \frac{1}{2},\ y = -\frac{1}{2}}$$

Partial fractions or Subst.

$$\frac{1}{G(y)} = \frac{y}{(1-2y)(1+2y)} = \frac{1/4}{1-2y} + \frac{-1/4}{1+2y}$$

or $\dfrac{y'dx}{G(y)} = -\dfrac{1}{8}\dfrac{du}{u}$ where $u = 1-4y^2$

(b) [85] • non-equil Sols $\dfrac{y'}{G(y)} = F(x)$

$$\frac{1}{4}\int\frac{dx}{1-2y} - \frac{1}{4}\int\frac{dx}{1+2y} = \int\left(6x^2-6x+18+\frac{-18}{1+x}\right)dx$$

$$\boxed{-\frac{1}{8}\ln|1-2y| - \frac{1}{8}\ln|1+2y| = 2x^3-3x^2+18x-18\ln|1+x|+c}$$

or

$$\boxed{-\frac{1}{8}\ln|1-4y^2| = 2x^3-3x^2+18x-18\ln|1+x|+c}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

4. (Linear Equations)

(a) Solve $2v'(t) = -64 + \frac{4}{t+1}v(t)$, $v(0) = 3$. Show all integrating factor steps.

(b) Using the answer $v(t)$ from (a), solve $y'(t) = v(t)$, $y(0) = 2$. Show all quadrature steps.

(a) [80]

$v' - \frac{2}{t+1}v = -32, \quad v(0) = 3$

$Q = e^{\int -\frac{2dt}{t+1}}$

$= e^{-2\ln|1+t|}$

$= (1+t)^{-2}$ wrong Q: -12

$\frac{(Qv)'}{Q} = -32$

$Qv = -32\int Q\,dt$

$Qv = -32\int (1+t)^{-2}dt$

$= -32\frac{(1+t)^{-1}}{-1} + c$

$v = c(1+t)^2 + 32(1+t)$

$3 = c + 32$

$-29 = c$

$\boxed{v = 32(1+t) - 29(1+t)^2}$

(b) [20]

$y' = 32(1+t) - 29(1+t)^2, \quad y(0) = 2$

$y = 2 + \int_0^t \left(32(1+t) - 29(1+t)^2\right)dt$

$$\boxed{\begin{aligned} y &= 2 + 32\left(\frac{(1+t)^2}{2} - \frac{1}{2}\right) + (-29)\left(\frac{(1+t)^3}{3} - \frac{1}{3}\right) \\ &= 16(1+t)^2 - \frac{29}{3}(1+t)^3 - 14 + \frac{29}{3} \\ &= -\frac{29}{3}t^3 - 13t^2 + 3t + 2 \end{aligned}}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name. KEY

5. (Stability)
(a) Draw a phase line diagram for the differential equation

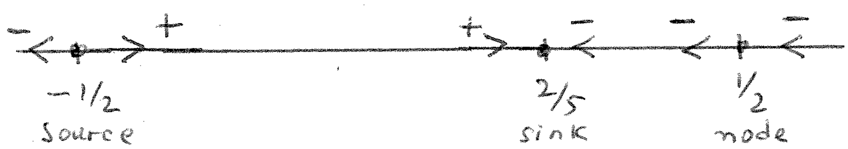
$$dx/dt = (2-5x)^3(1-2x)(1-4x^2).$$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers < and >).
(b) Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, stable, unstable. Show at least 10 threaded curves. A direction field is not required.

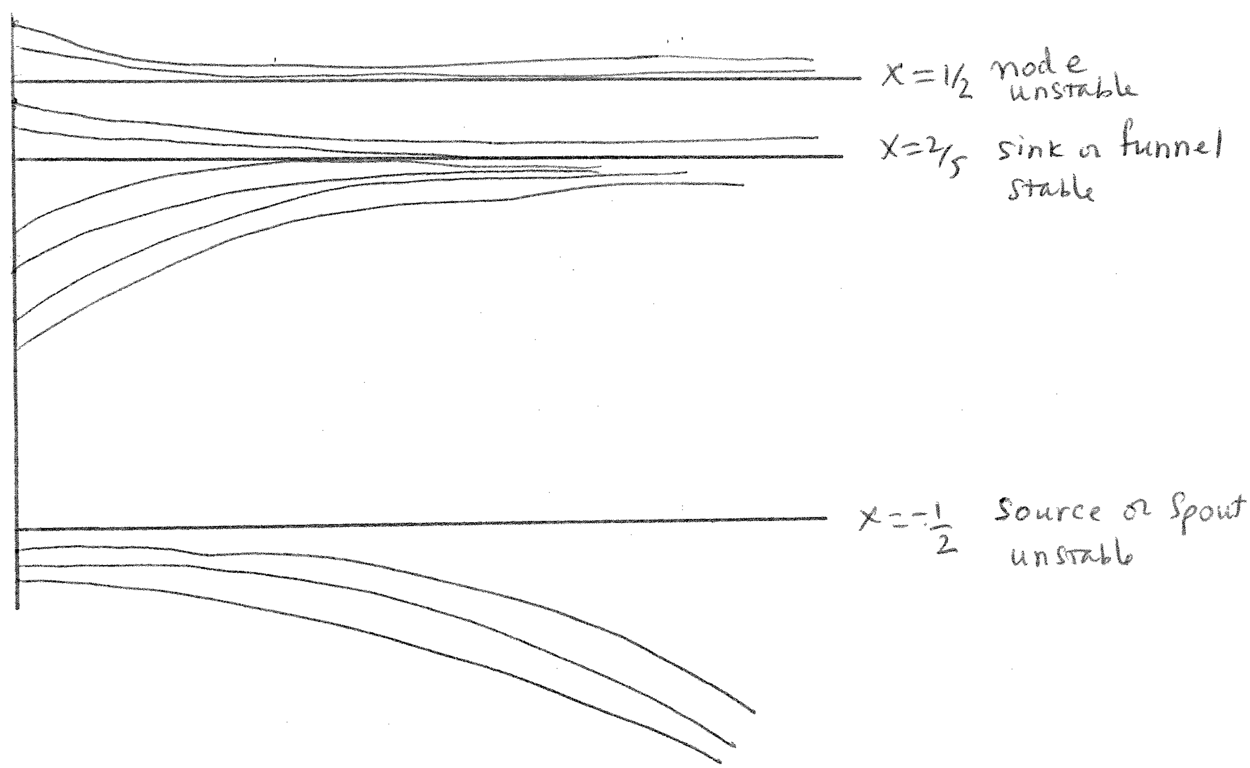
(a) $f(x) = (2-5x)^3(1-2x)^2(1+2x)$

roots $2/5,\ 1/2,\ -1/2$

$f(1) = (-)^3(-)^2(+) = (-)$
$f(0) = (+)^3(+)^2(+) = (+)$
$f(-1) = (+)^3(+)^2(-) = (-)$



(b)



Use this page to start your solution. Attach extra pages as needed, then staple.