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Differential Equations and Linear Algebra 2250-1

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Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **second entry on the fourth row** of the inverse matrix $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The only allowed use of Sarrus' rule is the 2×2 case.

$$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

☐ [50%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has infinitely many solutions [25%] and then for all such k display the solution formula for $\mathbf{x}$ [25%].

$$R = \begin{bmatrix} 2 & 1 & -12k & 0 \\ 2 & 6k & -2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1-6k \\ 0 \end{pmatrix}$$

☐ [50%] Ch3(c): Let A be a 30×31 matrix. Is it possible that $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x}$? Explain your answer fully.

☐ [50%] Ch3(d): Prove that if an $n \times n$ matrix is not invertible, then it has a zero eigenvalue.

(a) $C_{42} = \text{cofactor}(B, 2, 4)/\det(B) = \boxed{1}$. $\det(B) = 1 \cdot 1 \cdot \begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 4$, $\text{cof} = (-1)^6 \begin{vmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 4$

(b) $C = \text{aug}(R, f) \cong \left(\begin{array}{cccc|c} 2 & 1 & -12k & 0 & 0 \\ 0 & 6k-1 & 12k-2 & 0 & 1-6k \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \cong \left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 6k-1 & 0 & 0 & 1-6k \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$ One free var $\Leftrightarrow \boxed{1-6k=0}$

$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \boxed{\vec{x} = \begin{pmatrix} -t_1/2 \\ t_1 \\ 0 \\ t_2 \end{pmatrix}}$

(c) $\boxed{\text{NO}}$ rank ≤ 30, variable count $= 31$. Always one free var $\Leftrightarrow \infty$-many sols.

(d) A not invertible $\Leftrightarrow \det(A) = 0 \Leftrightarrow \det(A - \lambda I) = 0$ for $\lambda = 0$
$\Leftrightarrow \lambda = 0$ is an eigenvalue.

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, v3 in $\mathcal{R}^4$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 0 \end{pmatrix}.$$

☐ [60%] Ch4(b): Define V to be the set of all functions $f(x)$ defined on $a \le x \le b$ such that $f(0) + f(1) = 0$. Prove that V is a subspace of the vector space W of all functions $g(x)$ defined on $a \le x \le b$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for the column space of the 4×4 matrix A below. The reported basis must consist of columns of A.

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & -1 & 1 \\ 0 & 11 & 0 & 11 \\ 2 & 3 & -2 & 1 \end{pmatrix}.$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition of the fraction given below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{9x+1}{(x+1)^2(x+2)^2}$$

(a) independent $\Leftrightarrow$ rank $(\text{aug}(v_1, v_2, v_3)) = 3$. $\text{aug}(v_1, v_2, v_3) = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

$\cong \begin{pmatrix} -1 & 3 & 1 \\ 0 & 3 & 3 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ rank $= 2 \Rightarrow$ $\boxed{\text{dependent}}$

(b) The zero function satisfies $f(0)+f(1)=0$, so $\vec{0}$ in V. Given f_1, f_2 in V, Then $f = c_1 f_1 + c_2 f_2$ satisfies $f(0)+f(1) = c_1(f_1(0)+f_1(1)) + c_2(f_2(0)+f_2(1)) = 0$. By The subspace criterion, V is a subspace of W.

(c) $A \cong \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. pivot cols $= 1, 2$. $\boxed{\text{Basis} = \text{cols } 1, 2 \text{ of } A}$

(d) $\dfrac{25}{x+1} + \dfrac{-8}{(x+1)^2} + \dfrac{-25}{x+2} + \dfrac{-17}{(x+2)^2}$

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Ch5. (Linear Equations of Higher Order)

☐ [25%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations given below.

1.[10%] $10y'' + 11y' + 3y = 0$,
2.[15%] characteristic equation $(r-2)^2(r^2+2r+5)^2(r^2-4)^3 = 0$

☐ [25%] Ch5(b): Given $6x''(t) + 13x'(t) + 6x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 13$, $k = 6$, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate the model in a figure [5%].

☐ [50%] Ch5(c): Determine the **final form** of a trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv} + 50y'' + 625y = xe^{5x} + x\cos 5x + e^{-25x} + \sin 5x$$

☐ [25%] Ch5(d): Find the steady-state periodic solution for the equation

$$x'' + 2x' + 10x = \cos(3t).$$

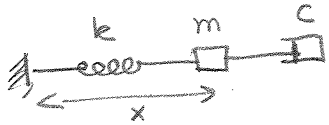
(a) 1. $y = c_1 e^{-t/2} + c_2 e^{-3t/5}$

2. $y = u_1 e^{2t} + u_2 e^{-2t} + u_3 e^{t}\cos 2t + u_4 e^{t}\sin 2t$

$u_1 = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4$, $u_2 = c_6 + c_7 t + c_8 t^2$,

$u_3 = c_9 + c_{10} t$, $u_4 = c_{11} + c_{12} t$

(b) $x(t) = c_1 e^{-2t/3} + c_2 e^{-3t/2}$, $\boxed{\text{overdamped}}$,



(c) $y = (d_1 + d_2 x)e^{5x} + d_3 e^{-25x} + (d_4 + d_5 x)\cos 5x + (d_6 + d_7 x)\sin 5x$

$r^4 + 50r^2 + 625 = (r^2+25)^2$ roots $\pm 5i, \pm 5i$ (double)

corrected $y = (d_1 + d_2 x)e^{5x} + d_3 e^{-25x}$ [same]

$+ x^2\left[(d_4 + d_5 x)\cos 5x + (d_6 + d_7 x)\sin 5x\right]$ [modified]

(d) $x_{ss} = \frac{1}{37}\cos(3t) + \frac{6}{37}\sin(3t)$, by undetermined coefficients.

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Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix A. Don't find eigenvectors!

$$A = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

☐ [35%] Ch6(b): Let A be a 2×2 matrix. Assume Fourier's method for A says that

$$\mathbf{x} = x_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

implies

$$A\mathbf{x} = 3x_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} - 7x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the matrix A.

☐ [35%] Ch6(c): Assume $\det(A - \lambda I) = \det(B - \lambda I)$ for two 5×5 matrices A, B. Let A have eigenvalues 1, 2, 6, a, b and let $5B$ have eigenvalues 10, $10 + 5i$, $10 - 5i$, c, d. Find the eigenvalues of A (determine a, b).

☐ [35%] Ch6(d): Display a 5×5 matrix A which has all entries above the diagonal positive and has eigenvalues $-1, -2, 2, 3, -5$ [15%]. Justify your claim by citing applicable determinant evaluation rules [20%].

(a) $\det(A - \lambda I) = (3-\lambda)(2-\lambda)((4-\lambda)^2 - 1) \Rightarrow$ eigenvalues $= \boxed{2, 3, 3, 5}$

(b) $AP = PD \Rightarrow A = PDP^{-1}$. $P = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$, $P^{-1} = -\frac{1}{6}\begin{pmatrix} -1 & -1 \\ -1 & 5 \end{pmatrix}$

$A = \begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix}\frac{1}{6} = \begin{pmatrix} 4/3 & 25/3 \\ 5/3 & -16/3 \end{pmatrix}$

(c) $\det(5B - \lambda I) = \det(5I)\det(B - \frac{\lambda}{5} I) = 5^5 \det(A - \frac{\lambda}{5} I)$, to match,

$\left\{ \frac{10}{5}, \frac{10+5i}{5}, \frac{10-5i}{5}, \frac{c}{5}, \frac{d}{5} \right\} = \{1, 2, 6, a, b\}$. Then $\boxed{a = 2+i,\ b = 2-i}$

(d) $A = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}$. By the triangular rule, A has eigenvalues = diag entries.

Staple this page to the top of all Ch6 work. Submit one package per chapter.

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Ch7. (Linear Systems of Differential Equations)

☐ [40%] Ch7(a): Let $x(t)$ and $y(t)$ be the amounts of salt in brine tanks A and B, respectively. Assume fresh water enters A at rate $r = 5$ gallons/minute. Let A empty to B at rate r, and let B empty at rate r. Assume the model

$$\begin{cases} x'(t) = -\dfrac{r}{50}x(t), \\ y'(t) = \dfrac{r}{50}x(t) - \dfrac{r}{100}y(t), \\ x(0) = 5, \quad y(0) = 10. \end{cases}$$

Find the maximum amount of salt ever in tank B.

☐ [60%] Ch7(b): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

☐ [40%] Ch7(c): Assume A is 4×4 and has eigenvalues -1, 5, $2 \pm \sqrt{5}i$. In the system $\mathbf{u}' = A\mathbf{u}$ where $\mathbf{u}(t)$ has components $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$ determine the list of atoms used to express the components in the general solution.

(a) The growth-decay theory implies $x(t) = 5e^{-t/10}$. Then $y' + \frac{1}{20}y = \frac{1}{2}e^{-t/10}$. Linear theory implies $y(t) = 20e^{-t/20} - 10e^{-t/10}$. Then $y' = 0$ when $\frac{1}{2}e^{-t/10} = e^{-t/20} - \frac{1}{2}e^{-t/10}$ or $e^{-t/10} = e^{-t/20}$. There is no critical pt for $t > 0$, so $\boxed{\max = y(0) = 10}$

(b) Eigenpairs $= \left(2, \begin{pmatrix}1\\1\\3\end{pmatrix}\right), \left(-5, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right), \left(-1, \begin{pmatrix}1\\1\\0\end{pmatrix}\right)$

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix}1\\1\\3\end{pmatrix} + c_2 e^{-5t}\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

(c) atoms $= e^{-t}, e^{5t}, e^{2t}\cos\sqrt{5}t, e^{2t}\sin\sqrt{5}t$

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Differential Equations and Linear Algebra 2250-1

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [30%] Ch10(a): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2s^2+6}{(s-1)(2s-2)(s-1)(1-s)}.$$

☐ [30%] Ch10(b): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{v} + 3x''' + 2x' = te^{4t} + e^t \sin 2t, \quad x(0) = x'(0) = x''(0) = x'''(0) = 0, \quad x^{iv}(0) = 2.$$

☐ [35%] Ch10(c): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 2x + 3y + e^t \sin t, \\ y'' &= 4x + 3y, \\ x(0) &= 0, \quad x'(0) = 1, \\ y(0) &= 2, \quad y'(0) = 0. \end{aligned}$$

☐ [35%] Ch10(d): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\frac{s}{(s^2+2s+5)} + \frac{s^2}{(s+1)^3} + \frac{2+s}{s^2+3s}.$$

☐ [35%] Ch10(e): Find $\mathcal{L}(f(t))$, given $f(t) = e^{-t}(\cos(2t) - 1)/t$.

(a) $\mathcal{L}(f) = \frac{-1}{(s-1)^2} + \frac{-2}{(s-1)^3} + \frac{-4}{(s-1)^4} = \mathcal{L}((-t - t^2 - 2t^3/3)e^t)$; apply Lerch's Thm.

(b) $(s^5 + 3s^3 + 2s)\mathcal{L}(x) = 2 + (s-4)^{-2} + \frac{2}{(s-1)^2+4}$. Divide.

(c) $\begin{cases} s^2\mathcal{L}(x) - 1 = 2\mathcal{L}(x) + 3\mathcal{L}(y) + \frac{1}{(s-1)^2+1} \\ s^2\mathcal{L}(y) - 2s = 4\mathcal{L}(x) + 3\mathcal{L}(y) \end{cases}$ $\quad \mathcal{L}(y) = \frac{\Delta_1}{\Delta}$ $\quad \Delta = \begin{vmatrix} s^2-2 & -3 \\ -4 & s^2-3 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} s^2-2 & 1+\frac{1}{(s-1)^2+1} \\ -4 & 2s \end{vmatrix}$

(d) $\mathcal{L}(f) = \mathcal{L}((-t)\,e^{-t}(\cos 2t - \sin 2t)) + \mathcal{L}((1 - 2t + t^2/2)e^{-t})$
$+ \mathcal{L}(\frac{1}{3}e^{-3t} + \frac{2}{3})$. Apply Lerch's Thm.

(e) $\mathcal{L}(tf(t)) = \mathcal{L}(\cos 2t - 1)\big|_{s\to s+1} = \left(\frac{s}{s^2+4} - \frac{1}{s}\right)\Big|_{s\to s+1}$

Then $\frac{d}{ds}\mathcal{L}(f) = -\left(\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right)$. Integrate, use $\mathcal{L}(f)\big|_{s=\infty} = 0$, get $\boxed{\mathcal{L}(f) = \ln\left(\frac{s+1}{\sqrt{(s+1)^2+4}}\right)}$

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