1. (ch4) Complete enough of the following to add to 100%.

(a) [100%] Let S be the vector space of all continuous functions defined on  $(-\infty, \infty)$ . Define V to be the set of all functions f(x) in S such that  $\int_0^1 f(x)(1+x)dx = 0$ . Prove that V is a subspace of S.

(b) [30%] Find a  $4 \times 5$  augmented matrix representing four equations for the constants a, b, c, d in the partial fractions decomposition for the fraction given below. To save time, **do not solve for** a, b, c, d!

$$\frac{x-2}{(x-1)^2(x^2-4x+8)}$$

(c) [70%] Solve for the unknowns a, b, c, d in the system of equations below by augmented matrix RREF methods, showing all details.

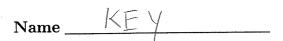
**Solution 1(a).** Use the subspace criterion: (a) Given f and g in V, write details to show f+g is in V; (b) Given f in V and k constant, write details to show kf is in V. Let h(x)=1+x, which is a function in S. Details for (a): Given  $\int_0^1 f(x)h(x)dx=0$  and  $\int_0^1 g(x)h(x)dx=0$ , add the equations to obtain the equation  $\int_0^1 (f(x)+g(x))h(x)dx=0$ . This finishes (a). Details for (b): Given  $\int_0^1 f(x)h(x)dx=0$  and k constant, multiply the equation by k and re-arrange factors to obtain the new equation  $\int_0^1 (kf(x))h(x)dx=0$ . This proves (b).

Solution 1(b). The decomposition can be

$$\frac{x-2}{(x-1)^2(x^2-4x+8)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c(x-2)+2d}{x^2-4x+8}$$

although there are other possibilities. Clear the fractions. Set x = 1 to get one equation for the constants. Choose 3 other values for x to obtain three other equations. Display the system of equations.

**Solution 1(c)**. The answer is 
$$\begin{pmatrix} 1+4t_-t_2 \\ -2t_1 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$



- 2. (ch5) Complete all parts.
  - (a) [30%] Given 4x''(t) + 8x'(t) + x(t) = 0, which represents a damped spring-mass system with m = 4, c = 8, k = 1, solve the differential equation [20%] and classify the answer as over-damped, critically damped or under-damped [10%].
  - (b) [70%] Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation  $x'' + 2x' + 8x = 10\cos(2t)$ .

(a) 
$$4r^2 + 8r + 1 = 0$$
 $r = -\frac{8}{8} + \frac{1}{8} \sqrt{64 - 16}$ 
 $= -1 + \frac{1}{8} \sqrt{48}$ 
 $= -1 + \frac{1}{2} \sqrt{3}$ 

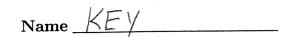
(b)  $r^2 + 2r + 8 = 0$ 
 $(r+1)^2 + 7 = 0$ 
 $r = -1 + \sqrt{7}\lambda$ 

No fixey mile.

 $\chi = d_1 \cos 2t + d_2 \sin 2t$ 
 $= -10 \cos (2t)$ 
 $= -10 \cos$ 

 $x_{ss} = \frac{3}{4} \cos(2t) + \frac{5}{4} \sin(2t)$  [=  $x_n + x_p \sin n_{eq} \exp terms$ 

Use this page to start your solution. Staple extra pages as needed.



- 3. (ch5) Complete all parts below.
  - (a) [75%] Determine for  $y^v 4y''' = xe^{2x} + x + x^3 + \sin x$  the corrected trial solution for  $y_p$  according to the method of undetermined coefficients. To save time, do not evaluate the undetermined coefficients (that is, do undetermined coefficient steps  $\boxed{1}$  and  $\boxed{2}$ , but skip steps  $\boxed{3}$  and  $\boxed{4}$ )! Undocumented details will be considered guessing, which earns no credit.
  - (b) [10%] Using the recipe for second order constant-coefficient differential equations, write out a basis for the solution space of the equation y'' + 2y' y = 0.
  - (c) [15%] Using the recipe for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is  $(r-1)^3(r^2-1)(r^2+4)^2=0$ .

Q RHS atoms = 
$$1, \times, \times^2, \times^3$$
  $e^{2x}$ ,  $xe^{2x}$  cos  $x$ ,  $xe^{2x}$  to  $e^{2x}$  thick  $e^{2x}$  thick  $e^{2x}$   $e^{2x}$  thick  $e^{2x}$   $e^{2x}$  thick  $e^{2x}$   $e^{2x}$  thick  $e^{2x}$   $e^{2x}$ 

(1) 
$$y^2 + 2r - 1 = 0$$
  $y = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t}$  by recipe  $(r+1)^2 - 2 = 0$  Bosis =  $\{2, y, 2, y\}$   $= -1 \pm \sqrt{2}$   $= -1 \pm \sqrt{2}$ 

(C) 
$$(r-1)^3(r-1)(r+1)(r^2+4)^2$$
  
 $y = u_1e^t + u_2e^t + u_3\cos(2t) + u_4\sin(2t)$   
 $u_1 = c_1 + c_2t + c_3t^2 + c_4t^3$  [mult = 4]  
 $u_2 = c_5$   
 $u_3 = c_6 + c_7t$   
 $u_4 = c_8 + c_4t$ 

Use this page to start your solution. Staple extra pages as needed.

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- 4. (ch6) Complete all of the items below.
  - (a) [30%] Find the eigenvalues of the matrix  $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ . To save time, **do not** find eigenvectors!
  - (b) [70%] Given  $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ , then there exists an invertible matrix P and a diagonal matrix D such that AP = PD. Find one possible column of P.

(a) 
$$det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1-1 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$
  
=  $(1-\lambda)(1-\lambda)((1-\lambda)^{2}+4)$   
[eigenvalue2 = 1,1,1=2i]

Description eigenvalue. Then are 4,6, one cold P is an eigenvector.

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0 & 1 & -1 & | & 0 \\
0 & 4 & 4 & | & 0
\end{pmatrix}$$

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0 & 0 & 5 & | & 0
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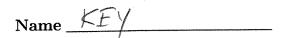
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\end{pmatrix}$$



5. (ch6) Complete all parts below.

Consider a given  $3 \times 3$  matrix A having three eigenpairs

$$6, \left(egin{array}{c}1\2\0\end{array}
ight); \quad 4, \left(egin{array}{c}-1\1\1\end{array}
ight); \quad 1, \left(egin{array}{c}1\-2\0\end{array}
ight).$$

(a) [50%] Display the general solution  $\mathbf{x}(t)$  of the system  $\mathbf{x}' = A\mathbf{x}$  in vector form.

(b) [20%] Write a matrix algebra formula for the matrix A of (a) above. To save time, do not evaluate anything.

(c) [30%] Describe precisely Fourier's simplification method for the equation  $\mathbf{y} = A\mathbf{x}$ , using the matrix A of (a) above.

(a) 
$$\vec{\chi}(t) = c_1(\frac{1}{2})e^{6t} + c_2(\frac{1}{1})e^{4t} + c_3(\frac{1}{2})e^{t}$$

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ A = PPP^{-1} \end{pmatrix}$$

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} \frac{1}{2} & \frac{1}{1} & -\frac{1}{2} \\ 0 & \frac{1}{1} & -\frac{2}{2} \end{pmatrix}$$

$$= \frac{1}{3} = \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} \left( \frac{1}{3} \right) + \frac{1}{4$$