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1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let S be the vector space of all continuous functions defined on $(-\infty, \infty)$. Define V to be the set of all functions $f(x)$ in S such that $\int_0^1 f(x)(1+x)dx = 0$. Prove that V is a subspace of S.

(b) [30%] Find a 4×5 augmented matrix representing four equations for the constants a, b, c, d in the partial fractions decomposition for the fraction given below. To save time, **do not solve for** a, b, c, d!

$$\frac{x-2}{(x-1)^2(x^2-4x+8)}$$

(c) [70%] Solve for the unknowns a, b, c, d in the system of equations below by augmented matrix RREF methods, showing all details.

$$\begin{array}{ccccccccc}
a & + & b & - & 2c & + & d & = & 1 \\
 & + & b & + & 2c & + & & = & 0 \\
a & + & 2b & + & & + & d & = & 1 \\
a & + & 3b & + & 2c & + & d & = & 1
\end{array}$$

Solution 1(a). Use the subspace criterion: (a) Given f and g in V, write details to show $f+g$ is in V; (b) Given f in V and k constant, write details to show kf is in V. Let $h(x) = 1+x$, which is a function in S. Details for (a): Given $\int_0^1 f(x)h(x)dx = 0$ and $\int_0^1 g(x)h(x)dx = 0$, add the equations to obtain the equation $\int_0^1 (f(x)+g(x))h(x)dx = 0$. This finishes (a). Details for (b): Given $\int_0^1 f(x)h(x)dx = 0$ and k constant, multiply the equation by k and re-arrange factors to obtain the new equation $\int_0^1 (kf(x))h(x)dx = 0$. This proves (b).

Solution 1(b). The decomposition can be

$$\frac{x-2}{(x-1)^2(x^2-4x+8)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c(x-2)+2d}{x^2-4x+8}$$

although there are other possibilities. Clear the fractions. Set $x = 1$ to get one equation for the constants. Choose 3 other values for x to obtain three other equations. Display the system of equations.

Solution 1(c). The answer is $\begin{pmatrix} 1+4t\_t_2 \\ -2t_1 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

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2. **(ch5)** Complete all parts.

(a) [30%] Given $4x''(t) + 8x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 8$, $k = 1$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [70%] Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 8x = 10\cos(2t)$.

(a) $4r^2+8r+1=0$

$r = \frac{-8}{8} \pm \frac{1}{8}\sqrt{64-16}$

$= -1 \pm \frac{1}{8}\sqrt{48}$

$= -1 \pm \frac{1}{2}\sqrt{3}$

$\boxed{x = c_1 e^{-t+\sqrt{3}t/2} + c_2 e^{-t-\sqrt{3}t/2}}$

$\boxed{\text{over-damped}}$

(b) $r^2+2r+8=0$

$(r+1)^2+7=0$

$r = -1 \pm \sqrt{7}i$

No fixup rule.

$x = d_1 \cos 2t + d_2 \sin 2t$

$(-4d_1 \cos 2t - 4d_2 \sin 2t) + 2(-2d_1 \sin 2t + 2d_2 \cos 2t) + 8(d_1 \cos 2t + d_2 \sin 2t)$

$= 10\cos(2t)$

$$\begin{cases} -4d_1 - 4d_2 + 8d_1 = 10 \\ -4d_2 - 4d_1 + 8d_2 = 0 \end{cases}$$

atoms matched left & right

$\begin{pmatrix} 4 & 4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -4 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \end{pmatrix}\frac{1}{32}$

$= \begin{pmatrix} 40/32 \\ 40/32 \end{pmatrix}$

$\boxed{x_{ss} = \frac{5}{4}\cos(2t) + \frac{5}{4}\sin(2t)}$ [$= x_h + x_p$ with neg exp terms removed]

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3. **(ch5)** Complete all parts below.

(a) [75%] Determine for $y^v - 4y''' = xe^{2x} + x + x^3 + \sin x$ the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps $\boxed{1}$ and $\boxed{2}$, but skip steps $\boxed{3}$ and $\boxed{4}$)! Undocumented details will be considered guessing, which earns no credit.

(b) [10%] Using the *recipe* for second order constant-coefficient differential equations, write out a **basis** for the solution space of the equation $y'' + 2y' - y = 0$.

(c) [15%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r-1)^3(r^2-1)(r^2+4)^2 = 0$.

(a) RHS atoms $= 1, x, x^2, x^3, \quad e^{2x}, xe^{2x}, \quad \cos x, \sin x$

trial sol $= y$

$= y_1 + y_2 + y_3$

$$\boxed{\begin{cases} y_1 = d_1 + d_2 x + d_3 x^2 + d_4 x^3 \\ y_2 = (d_5 + d_6 x)e^{2x} \\ y_3 = d_7 \cos x + d_8 \sin x \end{cases}}$$

$r^5 - 4r^3 = r^3(r-2)(r+2)$

Duplicates for $r = 0$ (mult $s = 3$)

$r = 2$ (mult $s = 1$)

Replace y_1 by $x^3 y_1$, y_2 by xy_2. Don't fix y_3.

$\boxed{\text{correct trial sol} = x^3 y_1 + xy_2 + y_3}$

(b) $r^2 + 2r - 1 = 0$

$(r+1)^2 - 2 = 0$

$r = -1 \pm \sqrt{2}$

$y = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t}$ by recipe

Basis $= \{\partial_{c_1} y, \partial_{c_2} y\}$

$= \boxed{\{e^{-t+\sqrt{2}t}, e^{-t-\sqrt{2}t}\}}$

(c) $(r-1)^3(r-1)(r+1)(r^2+4)^2$

$$\boxed{\begin{array}{ll} y = u_1 e^t + u_2 e^{-t} + u_3 \cos(2t) + u_4 \sin(2t) & \\ u_1 = c_1 + c_2 t + c_3 t^2 + c_4 t^3 & [\text{mult} = 4] \\ u_2 = c_5 & [\text{mult} = 1] \\ u_3 = c_6 + c_7 t & [\text{mult} = 2] \\ u_4 = c_8 + c_9 t & \end{array}}$$

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4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [70%] Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$, then there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Find one possible column of P.

(a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 & 0 \\ 0 & 1-\lambda & -2 & 1 \\ 0 & 0 & 1-\lambda & -2 \\ 0 & 0 & 2 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)(1-\lambda)((1-\lambda)^2+4)$

$\boxed{\text{eigenvalues} = 1, 1, 1 \pm 2i}$

(b) $\lambda = 1$ is an eigenvalue. Others are 4, 6. One col of P is an eigenvector.

$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{cases} x = t_1 \\ y = 0 \\ z = 0 \end{cases}$ gen sol

eigenvector $= \partial_{t_1} = \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$.

Eigen pair $= \left(1, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$

Others are $\left(4, \begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}\right)$

and $\left(6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$.

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5. **(ch6)** Complete all parts below.

Consider a given 3×3 matrix A having three eigenpairs

$$6, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad 1, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

(a) [50%] Display the general solution $\mathbf{x}(t)$ of the system $\mathbf{x}' = A\mathbf{x}$ in vector form.

(b) [20%] Write a matrix algebra formula for the matrix A of (a) above. To save time, do not evaluate anything.

(c) [30%] Describe precisely Fourier's simplification method for the equation $\mathbf{y} = A\mathbf{x}$, using the matrix A of (a) above.

(a) $$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{t}$$

(b) $AP = PD$ $\qquad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix}$

$$\boxed{A = PDP^{-1}}$$

(c) $$\vec{x} = x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{y} = A\vec{x} = 6x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + 4x_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

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