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1. (rref)
Determine a, b such that (1) the system has no solution and (2) the system has infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ 5x & + & 10y & + & 2z & = & 2 \\ 6x & + & 2ay & + & bz & = & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 5 & 10 & 2 & 2 \\ 6 & 2a & b & 2 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 2a-12 & b-6 & -4 \end{array}\right) \quad \begin{array}{l} \text{combo}(1,2,-5) \\ \text{combo}(1,3,-6) \end{array}$$

$$\cong \left(\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2a-12 & b-6 & -4 \end{array}\right) \quad \text{mult}(2,-1/3)$$

$$\cong \left(\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2a-12 & 0 & -4+6-b \end{array}\right) \quad \begin{array}{l} \text{combo}(2,1,-1) \\ \text{combo}(2,3,6-b) \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2a-12 & 0 & 2-b \end{array}\right) \quad \begin{array}{l} x = \text{lead} \\ z = \text{lead} \end{array}$$

If $2a-12 \neq 0$, Then unique sol.

If $2a-12=0$ and $2-b \neq 0$, Then No sol.	$a=6, b\neq 2$
If $2a-12=0$ and $2-b=0$, Then ∞-many sols	$a=6, b=2$

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2. (vector spaces) Do **two** of the following but not three.

(a) [50%] Let V be the vector space of functions $f(t) = c_1 + c_2 e^t + c_3 t e^t + c_4(1 - e^t)$, for all values of c_1, c_2, c_3, c_4. Report a basis for V. Don't justify.

(b) [50%] Prove by means of the subspace criterion (Theorem 1, Edwards-Penney) that the set S of all fixed vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ with $v_2 + v_3 = 0$ is a subspace.

(c) [50%] Find a basis of 3-vectors for the solution space of the system of equations

$$\begin{aligned} x + y - 4z &= 0, \\ x + 3y - 2z &= 0, \\ 2y + 2z &= 0, \end{aligned}$$

(a) Basis is contained in $\{\partial_{c_1} f, \partial_{c_2} f, \partial_{c_3} f, \partial_{c_4} f\} = \{1, e^t, te^t, 1-e^t\}$
But $1-e^t$ is a l.c. of the others. $\boxed{\text{Basis} = \{1, e^t, te^t\}}$

(b) write the restriction equation as $\begin{cases} v_3 + v_2 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \iff \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff A\vec{v} = \vec{0}$ where $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $S = \{\vec{v} \text{ in } \mathbb{R}^3 : A\vec{v} = \vec{0}\}$

(1) $\vec{v}_1, \vec{v}_2$ in $S \Rightarrow A\vec{v}_1 = \vec{0},\ A\vec{v}_2 = \vec{0}$

$$\begin{aligned} \Rightarrow A(\vec{v}_1 + \vec{v}_2) &= A\vec{v}_1 + A\vec{v}_2 \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

$\Rightarrow \vec{v}_1 + \vec{v}_2$ in S.

(2) $\vec{v}$ in S and k = constant $\Rightarrow A\vec{v} = 0$ and k = constant

$$\begin{aligned} \Rightarrow A(k\vec{v}) &= k(A\vec{v}) \\ &= k\vec{0} \\ &= \vec{0} \end{aligned}$$

$\Rightarrow k\vec{v}$ in S.

Proof is complete.

(c) $\left(\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Gen. Sol. is $\begin{cases} x = 5t_1 \\ y = -t_1 \\ z = t_1 \end{cases}$ Basis $= \{\partial_{t_1} \vec{x}\} = \boxed{\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}}$

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3. (independence) Do either (a) or (b) but not both.

(a) [100%] Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 5 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 0 \end{pmatrix}$. State and apply a test that shows $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are dependent.

(b) [100%] Let matrix D be given and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be vectors with $D\mathbf{a}$, $D\mathbf{b}$, $D\mathbf{c}$ independent. Prove that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are independent. Don't do this problem if you did (a)!

(a) $\vec{u}, \vec{v}, \vec{w}$ are independent in $\mathbb{R}^3 \iff$ rref(aug($\vec{u}, \vec{v}, \vec{w}$)) has 3 nonzero rows

$\iff$ rank(rref(aug($\vec{u}, \vec{v}, \vec{w}$))) $= 3$

$\vec{u}, \vec{v}, \vec{w}$ are independent in $\mathbb{R}^3 \iff$ det(aug($\vec{u}, \vec{v}, \vec{w}$)) $\neq 0$

$$A = \text{aug}(\vec{u}, \vec{v}, \vec{w}) = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 5 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

second test does not apply because A is not 3×3.

$$\cong \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 1 & 5 & 4 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rref found; $\boxed{\text{2 nonzero rows} \iff \text{dependent}}$

(b) Form the equation

$$c_1 \vec{a} + c_2 \vec{b} + c_3 \vec{c} = \vec{0}$$

Multiply by matrix D:

$$c_1 D\vec{a} + c_2 D\vec{b} + c_3 D\vec{c} = \vec{0}$$

By independence of $D\vec{a}$, $D\vec{b}$, $D\vec{c}$ we have $c_1 = c_2 = c_3 = 0$.

Therefore, $\{\vec{a}, \vec{b}, \vec{c}\}$ is an independent set.

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4. **(determinants and elementary matrices)**
Assume given two invertible 3×3 matrices A, B. Let elementary matrices E_1, E_2, E_3 be given, with E_1 a swap, E_2 a combination and E_3 a multiply, with multiplier $1/3$, and assume $E_1 E_2 B = E_3 A$. Explain precisely why $\det(3BA^{-1}) = -9$.

(1) $\det(3BA^{-1}) = \det((3I)(B)(A^{-1}))$

$= \det(3I)\det(BA^{-1})$

use prod rule freely: $\det(CD) = \det(C)\det(D)$

$\det(E_1 E_2 BA^{-1}) = \det(E_3)$

(2) $\det(E_1)\det(E_2)\det(BA^{-1}) = \det(E_3)$ use given $E_1E_2B = E_3A$

$$\begin{cases} \det(E_1) = -1 & \text{swap} \\ \det(E_3) = 1/3 & \text{mult} \\ \det(E_2) = 1 & \text{combo} \end{cases}$$

$\therefore (-1)(1)\det(BA^{-1}) = 1/3$

$\det(3I) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 27$

$\det(3BA^{-1}) = \det(3I)\det(BA^{-1})$ from above (1) and (2)

$= (27)\left(-\frac{1}{3}\right)$

$= -9$

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5. (inverses and Cramer's rule)

(a) [75%] Determine all values of x for which A^{-1} fails to exist: $A = \begin{pmatrix} 2 & 2 & 0 & x \\ 2 & 0 & -3 & 0 \\ 0 & x & 1 & 1 \\ 0 & 2x & 1 & 0 \end{pmatrix}$.

(b) [25%] State two determinant rules which follow from the four rules *triangular, swap, combo, mult.* Don't give any proofs.

(a) A^{-1} exists $\iff \det(A) \neq 0$. Fails when $\det(A) = 0$

$$\det(A) = (2)(1)\begin{vmatrix} 0 & -3 & 0 \\ x & 1 & 1 \\ 2x & 1 & 0 \end{vmatrix} + (2)(-1)\begin{vmatrix} 2 & 0 & x \\ x & 1 & 1 \\ 2x & 1 & 0 \end{vmatrix}$$

cofactor expansion on Col = 1

$$= (2)(1)(-1)\begin{vmatrix} 0 & -3 \\ 2x & 1 \end{vmatrix} + (2)(-1)\left(2\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + x\begin{vmatrix} x & 1 \\ 2x & 1 \end{vmatrix}\right)$$

$$= -(2)(6x) - 2(-2 - x^2)$$

$$= -12x + 4 + 2x^2$$

The two values of x satisfy

$$2x^2 - 12x + 4 = 0$$
$$x^2 - 6x + 2 = 0$$
$$x = \frac{6 \pm \sqrt{36 - 8}}{2}$$

(b) Rule 1. If A has a row of zeros, then $\det(A) = 0$

Rule 2 If A has 2 duplicate rows, then $\det(A) = 0$

Both rules may be stated for columns also, due to $\det(A^T) = \det(A)$. Finally, the rules can be stated for both rows and cols, but only one case is expected.

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