Applied Differential Equations 2250-1
Midterm Exam 1
Wednesday, 28 September 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for \( y(x) \) in the equation 
\[
y' = xe^{-x^2/2} + \sec^2 x + \tan^2 x + \frac{48x^5}{1 + 4x^3}.
\]

\[
\int y' \, dx = \int xe^{-x^2/2} \, dx + \int \left( \sec^2 x + (\sec^2 x - 1) \right) \, dx + \int \frac{48x^5}{1 + 4x^3} \, dx
\]

\[
y = uv - \int v \, du + \int (2 \sec^2 x - 1) \, dx + \int \left( \frac{\text{quotient}}{\text{remainder}} \right) \, dx
\]

\[
\begin{aligned}
\frac{u}{dv} &= xe^{-x^2/2} \, dx \\
1 + \tan \theta &= \sec \theta
\end{aligned}
\]

\[
y = \frac{x e^{-x^2/2}}{-1/2} - \int \frac{e^{-x^2/2}}{-1/2} \, dx + 2 \tan x - x + \int \frac{12x^2}{1 + 4x^3} \, dx + \int \frac{12x^2}{48x^5 + 12x^2} \, dx
\]

\[
y' = -2xe^{-x^2/2} - 4e^{-x^2/2} + 2 \tan x - x + 4x^3 - \ln|1 + 4x^3| + C
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Separable Equation Test)

The problem

\[ y' = x \left( \cos(x)ye^{3y} - \cos(x)\cos(y) + e^{3x}x^{2/5}ye^{3y} \right) - e^{3x}x^{7/5}\cos(y) \]

may or may not be separable. If it is, then write formulae for \( F, G \) which decompose the problem as \( y' = F(x)G(y) \). Otherwise, explain in detail why it fails to be separable. Do not solve for \( y \! \)!

The integrals

\[ f(x) = \int (x \cos(x)ye^{3y} - \pi \cos(x)\cos(y) + e^{3x}x^{2/5}ye^{3y} - e^{3x}x^{7/5}\cos(y) \text{ with } x_0 = \pi, y_0 = 0 \]

\[ c_0 = \pi \]

\[ f(x, y) = -x \cos(x) - x^{7/5}e^{3x} + \pi \cos(x)\cos(y) + e^{3x}x^{2/5}ye^{3y} - e^{3x}x^{7/5}\cos(y) \]

\[ F(x) = \int f(x) = -\pi \cos(x) + \pi \cos(y) + e^{3x}x^{2/5}ye^{3y} - e^{3x}x^{7/5}\cos(y) \]

\[ c = \pi e^{3x} \]

By the test, \[ It's \text{ separable } \]

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3. (Solve a Separable Equation)

Given \( yy' = \frac{x^3 + x}{1 + x} (4 - y^2) \),

(a) Find all equilibrium solutions,
(b) Find the non-equilibrium solution in implicit form.

Do not solve for \( y \) explicitly.

\[
F(x) = \frac{x^3 + x}{1 + x} = x^2 - x + 2 + \frac{-2}{1 + x}
\]

\[
G(y) = \frac{4 - y^2}{y}
\]

\[ y' = F(x) \cdot G(y) \quad \text{Separable Type DE} \]

\[ 15 \]

(a) equil sols

Solve \( G(y) = 0 \) to get \( y = 2, y = -2\)

\[ 85 \]

(b) non-equil sols

Solve \( \frac{y'}{G(y)} = F(x) \) by quadrature,

\[
\int \frac{y y'}{4 - y^2} \, dx = \left( x^2 - x + 2 + \frac{2}{1 + x} \right) \, dx
\]

\[
-\frac{1}{2} \int \frac{du}{u} = \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2 \ln(1 + y) + C
\]

\[
-\frac{1}{2} \ln(4 - y^2) = \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2 \ln(1 + x) + C
\]

\[
\begin{align*}
\frac{u}{4 - y^2} & = -2yy' \\
\frac{du}{u} & = -2yy' \, dx
\end{align*}
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
4. (Linear Equations)

(a) Solve $3v'(t) = -96 + \frac{6}{t+1}v(t), v(0) = 2$. Show all integrating factor steps.

(b) Using the answer $v(t)$ from (a), solve $y'(t) = v(t), y(0) = 3$. Show all quadrature steps.

(a) \[
\begin{align*}
3v' - \frac{2}{t+1}v &= -\frac{96}{3} \\
\int \frac{-2}{t+1} &= -2 \ln |t+1| \\
&= (t+1)^{-2}
\end{align*}
\]

\[
\begin{align*}
\frac{(t+1)^{-2}v'}{(t+1)^{-2}} &= -32 \\
((t+1)^{-2}v)' &= -32 (t+1)^{-2} \\
(t+1)^{-2}v &= c + 32 (t+1)^{-1}
\end{align*}
\]

\[
\begin{align*}
v &= c (t+1)^2 + 32 (t+1) \\
2 &= c + 32 \\
v &= -30 (t+1)^2 + 32 (t+1)
\end{align*}
\]

(b) \[
\begin{align*}
y' &= -30 (t+1)^2 + 32 (t+1) \\
y &= -10 (t+1)^3 + 16 (t+1)^2 + c \\
3 &= -10 + 16 + c \\
-3 &= c \\
y &= -10 (t+1)^3 + 16 (t+1)^2 - 3
\end{align*}
\]

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5. (Stability)
   (a) Draw a phase line diagram for the differential equation

   \[ \frac{dx}{dt} = (3 - 2x)^3(3 + x)(9 - x^2). \]

   Expected in the diagram are equilibrium points and signs of \( x' \) (or flow direction markers < and >).

   (b) Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, stable, unstable. Show at least 10 threaded curves. A direction field is not needed nor required.

   (a) \( f(x) = (3 - 2x)^3(3 + x)^2(3 - x) \)

   roots: \( 3/2, -3, 3 \)

   \( f'(x) = (-)(+)(-) = (+) \)

   

   (b) Use this page to start your solution. Attach extra pages as needed; then staple.