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Applied Differential Equations 2250-1
Midterm Exam 1
Wednesday, 28 September 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for $y(x)$ in the equation $y' = xe^{-x/2} + \sec^2 x + \tan^2 x + \dfrac{48x^5}{1+4x^3}$.

$$\int y'\,dx = \int x e^{-x/2}dx + \int \left(\sec^2 x + (\sec^2 x - 1)\right)dx + \int \frac{48x^5\,dx}{1+4x^3}$$

$$y = uv - \int v\,du + \int (2\sec^2 x - 1)\,dx + \int \left(\text{quotient} + \frac{\text{remainder}}{1+4x^3}\right)dx$$

$u = x$, $dv = e^{-x/2}dx$

$1+\tan^2\theta = \sec^2\theta$

$$\begin{array}{r|l} & 12x^2 \leftarrow \text{quotient} \\ 1+4x^3 & 48x^5 \\ \uparrow & 48x^5 + 12x^2 \\ \text{divisor} & -12x^2 \\ & \uparrow \\ & \text{remainder} \end{array}$$

$$y = \frac{xe^{-x/2}}{-1/2} - \int \frac{e^{-x/2}}{-1/2}dx + 2\tan x - x + \int 12x^2dx + \int \frac{-12x^2}{1+4x^3}dx$$

$$\boxed{y = -2xe^{-x/2} - 4e^{-x/2} + 2\tan x - x + 4x^3 - \ln|1+4x^3| + c}$$

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2. (Separable Equation Test)
The problem

$$y' = x\left(\cos(x)ye^{3y} - \cos(x)\cos(y) + e^{3x}x^{2/5}ye^{3y}\right) - e^{3x}x^{7/5}\cos(y)$$

may or may not be separable. If it is, then write formulae for F, G which decompose the problem as $y' = F(x)G(y)$. Otherwise, explain in detail why it fails to be separable. **Do not solve** for y!

$$f = x\cos(x)\,y\,e^{3y} - x\cos x\cos y + e^{3x}e^{3y}x^{7/5}y - e^{3x}x^{7/5}\cos y$$

choose $x_0 = \pi$, $y_0 = 0$

$$f(x_0,y_0) = \pi\cos\pi\,(0)e^0 - \pi\cos\pi\cos 0 + e^0 e^{3\pi}\pi^{7/5}\cdot 0 - e^{3\pi}\pi^{7/5}\cos 0$$
$$= \pi - e^{3\pi}\pi^{7/5}$$
$$\neq 0$$

$$F(x) = \frac{f(x,0)}{f(x_0,y_0)} = \frac{-x\cos x - x^{7/5}e^{3x}}{c_1} \qquad c_1 = \pi - e^{3\pi}\pi^{7/5}$$

$$G(y) = f(\pi,y) = -\pi y e^{3y} + \pi\cos y + e^{3\pi}\pi^{7/5}ye^{3y} - e^{3\pi}\pi^{7/5}\cos y$$

$$F(x)G(y) = \frac{-1}{c_1}\left(x\cos x + x^{7/5}e^{3x}\right)\left(-c_1 y e^{3y} + c_1\cos y\right)$$
$$= \frac{c_1}{c_1}\left(xy\cos x\,e^{3y} + x^{7/5}y\,e^{3x}e^{3y} - x\cos x\cos y - x^{7/5}e^{3x}\cos y\right)$$
$$= (1)\,f(x,y)$$
$$= f(x,y)$$

By The Test, $\boxed{\text{It's separable}}$

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3. (Solve a Separable Equation)

Given $yy' = \dfrac{x^3+x}{1+x}(4-y^2)$,

(a) Find all equilibrium solutions,

(b) Find the non-equilibrium solution in implicit form.

Do not solve for y explicitly.

$$F(x) = \frac{x^3+x}{1+x} = x^2 - x + 2 + \frac{-2}{1+x}$$

$$\begin{array}{r} x^2 - x + 2 \\ 1+x \,\overline{)\, x^3 + x} \\ \underline{x^3 + x^2} \\ -x^2 + x \\ \underline{-x^2 - x} \\ 2x \\ \underline{2x+2} \\ -2 \end{array}$$

$$G(y) = \frac{4-y^2}{y}$$

$y' = F(x)G(y)$ Separable Type DE

[15] (a) equil sols

Solve $G(y)=0$ to get $\boxed{y=2,\ y=-2}$

[85] (b) non-equil sols

Solve $\dfrac{y'}{G(y)} = F(x)$ by quadrature.

$$\int \frac{yy'dx}{4-y^2} = \int\left(x^2 - x + 2 - \frac{2}{1+x}\right)dx$$

$$-\frac{1}{2}\int \frac{du}{u} = \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2\ln|1+x| + C$$

$$\boxed{\begin{aligned} u &= 4-y^2 \\ du &= -2yy'dx \end{aligned}}$$

$$\boxed{-\frac{1}{2}\ln|4-y^2| = \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2\ln|1+x| + c}$$

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4. (Linear Equations)

(a) Solve $3v'(t) = -96 + \frac{6}{t+1} v(t)$, $v(0) = 2$. Show all integrating factor steps.

(b) Using the answer $v(t)$ from (a), solve $y'(t) = v(t)$, $y(0) = 3$. Show all quadrature steps.

(a) [80]

$$v' - \frac{2}{t+1} v = -\frac{96}{3}$$ Div by 3, std form

$$e^{\int -2/(t+1)} = e^{-2 \ln|t+1|} = (t+1)^{-2}$$ integ factor; wrong factor $= -12$

$$\frac{\left((t+1)^{-2} v\right)'}{(t+1)^{-2}} = -32$$ quadr form

$$\left((t+1)^{-2} v\right)' = -32 (t+1)^{-2}$$

$$(t+1)^{-2} v = c + 32 (t+1)^{-1}$$

$$v = c(t+1)^2 + 32(t+1)$$

$$2 = c + 32$$ use $v(0) = 2$

$$\boxed{v = -30(t+1)^2 + 32(t+1)}$$

(b) [20]

$$y' = -30(t+1)^2 + 32(t+1)$$

$$y = -10(t+1)^3 + 16(t+1)^2 + c$$

$$3 = -10 + 16 + c$$

$$-3 = c$$

$$\boxed{\begin{aligned} y &= -10(t+1)^3 + 16(t+1)^2 - 3 \\ &= -10t^3 - 14t^2 + 2t + 3 \end{aligned}}$$

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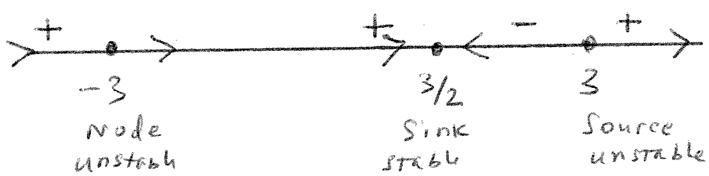
5. (Stability)
(a) Draw a phase line diagram for the differential equation

$$dx/dt = (3-2x)^3(3+x)(9-x^2).$$

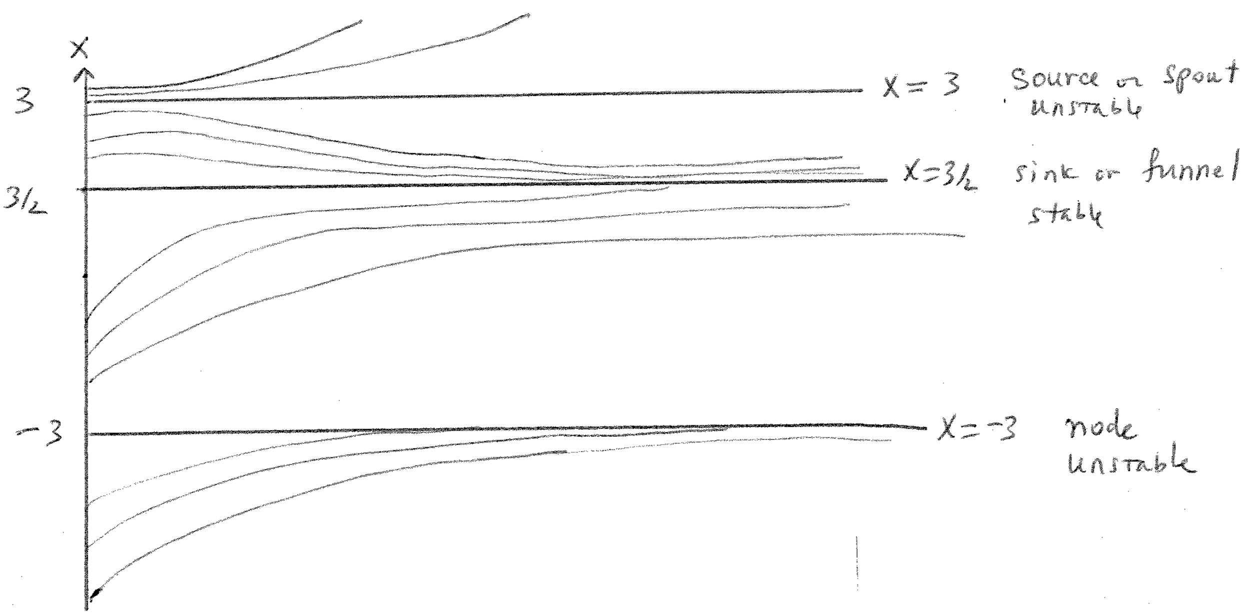
Expected in the diagram are equilibrium points and signs of x' (or flow direction markers < and >).
(b) Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, stable, unstable. Show at least 10 threaded curves. A direction field is not needed nor required.

(a) $f(x) = (3-2x)^3(3+x)^2(3-x)$ $\qquad f(4) = (-)(+)(-) = (+)$

roots $3/2, -3, 3$



(b)



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