Name KEY **2250-2**

Differential Equations and Linear Algebra 2250-2

Final Exam 10:30am 13 Dec 2004

Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **first two entries along the first row** of the inverse matrix B^{-1} by the formula $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of 3×3 Sarrus' rule is disallowed (2×2 use is allowed).

$$B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

☐ [50%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has a unique solution [25%] and then for all such k display the solution formula for $\mathbf{x}$ [25%].

$$R = \begin{bmatrix} 4 & 2 & -k \\ 2 & k & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1-k \\ 0 \end{pmatrix}$$

☐ [50%] Ch3(c): Let A be an 11×11 triangular matrix with all diagonal entries equal to e^{-2}. Prove that $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b}$.

(a) $\det B = 1$ ans $= 1, 0$

(b) $\det R = 4(4k-4)$ unique for $k-1 \neq 0$.

$$\left(\begin{array}{ccc|c} 4 & 2 & -k & 0 \\ 2 & k & -2 & 1-k \\ 0 & 0 & 4 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 0 & 2-2k & 0 & -2+2k \\ 2 & k & 0 & 1-k \\ 0 & 0 & 1 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1-2k \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\boxed{y=1, z=0, x=\frac{1}{2}-k}$

(c) Since $\det(A) = (e^{-2})^{11} \neq 0$, $Ax=b$ is solved by $x=A^{-1}b$, because then A^{-1} exists.

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Name KEY **2250-2**

Differential Equations and Linear Algebra 2250-2

Final Exam 10:30am 13 Dec 2004

Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v3}$ in $\mathcal{R}^3$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

☐ [60%] Ch4(b): Let A be a 13×13 matrix. Assume V is the set of all vectors $\mathbf{x}$ such that $A^3\mathbf{x} + 2A^2\mathbf{x} + \mathbf{x} = \mathbf{0}$. Prove that V is a subspace of $\mathcal{R}^{13}$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for the solution space of $A\mathbf{x} = \mathbf{0}$:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & -3 & 1 \\ 2 & 2 & -4 & 1 \end{bmatrix}$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{x^2+3x+1}{(x-1)^2(x+2)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2} + \frac{d}{(x+2)^2}$$

(a) rref(aug(v_1, v_2, v_3)) $= I \Leftrightarrow v_1, v_2, v_3$ indep.

$$\begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ -2 & 1 & 3 \end{pmatrix} \cong \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 7 & 7 \end{pmatrix} \cong \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \boxed{\text{dep}}$$

(b) Let $B = A^3 + 2A^2 + I$. Then $V = \{\vec{x} : B\vec{x} = \vec{0}\}$ and by the Superposition Thm V is a subspace.

(c)
$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis $= \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(d) $b = 5/9$, $d = -1/9$, $a = \frac{17}{27}$, $c = -\frac{2}{27}$

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Name ~~KEY~~ **2250-2**

Differential Equations and Linear Algebra 2250-2

Final Exam 10:30am 13 Dec 2004

Ch5. (Linear Equations of Higher Order)

☐ [30%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

$$\textbf{1.}[15\%]\quad (r^2+2r)(r+2)^2=0,$$
$$\textbf{2.}[15\%]\quad (r+2)^2(r^2+3)^2(r^2-4)=0$$

☐ [30%] Ch5(b): Given $4x''(t)+8x'(t)+3x(t)=0$, which represents a damped spring-mass system with $m=4$, $c=8$, $k=3$, solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

☐ [40%] Ch5(c): Determine the **final form** of a trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv}-8y''+16y=x^2e^{2x}+\sin 2x+e^{-2x}$$

☐ [30%] Ch5(d): Find the steady-state periodic solution for the equation

$$x''+2x'+5x=7\cos(3t).$$

(a) $r(r+2)^3=0$; $\boxed{y_h}=c_1+(c_2+c_3x+c_4x^2)e^{-2x}$

$(r-2)(r+2)^3(r^2+3)^2=0$; $\boxed{y_h}=c_1e^{2x}+(c_2+c_3x+c_4x^2)e^{-2x}$
$+(c_5+c_6x)\cos\sqrt{3}x+(c_7+c_8x)\sin\sqrt{3}x$

(b) $4r^2+8r+3=0$, $(2r+3)(2r+1)=0$: $\boxed{x=c_1e^{-3t/2}+c_2e^{-t/2}}$ $\boxed{\text{over damped}}$

(c) $\boxed{y_p}=(d_1+d_2x+d_3x^2)e^{2x}\cdot x^2$
$+(d_4e^{-2x})x^2+d_5\cos 2x+d_6\sin 2x$

$r^4-8r^2+16=(r^2-4)^2=(r-2)^2(r+2)^2$

(d) $x=A\cos 3t+B\sin 3t$
$x'=-3A\sin 3t+3B\cos 3t$
$x''=-9A\cos 3t-9B\sin 3t$

$\boxed{x=\dfrac{-14\cos 3t+21\sin 3t}{26}}$

$7\cos 3t=x''+2x'+5x$
$=(-9A+6B+5A)\cos 3t+(-9B-6A+5B)\sin 3t$

$\begin{cases}-4A+6B=7\\-6A-4B=0\end{cases}$ $\quad B=-3A/2$ $\quad -4A+6(-\frac{3A}{2})=7$ $\quad \boxed{A=-\frac{7}{13},\ B=\frac{21}{26}}$

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Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:30am 13 Dec 2004

Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

☐ [35%] Ch6(b): Let A be a 3×3 matrix with eigenpairs

$$(1, \mathbf{v}_1), \quad (2, \mathbf{v}_2), \quad (3, \mathbf{v}_3).$$

Let $P = \mathbf{aug}(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2)$. Find a diagonal matrix D such that $AP = PD$ [15%]. Justify your claims [20%].

☐ [35%] Ch6(c): Assume $\det(A - \lambda I) = \det(B - \lambda I)$ for two 7×7 matrices A, B. Let A have eigenvalues 1, 2, 3, $1 \pm i$, $2 \pm \sqrt{3}i$. Find the eigenvalues of $B + I$, where I is the identity matrix.

☐ [35%] Ch6(d): Give an example of a 3×3 matrix C which has eigenvalues 5, 5, 5 and three independent eigenvectors [15%]. Justify your claim [20%].

(a) $\boxed{1, 1, 1, 6}$

(b) $\boxed{D = \mathrm{diag}(3, 1, 2)}$ $PD = AP$ means $\begin{cases} 3v_3 = Av_3 \\ v_1 = Av_1 \\ 2v_2 = Av_2 \end{cases}$

(c) $\boxed{2, 3, 4, 2 \pm i, 3 \pm \sqrt{3}i}$

(d) $\boxed{C = \mathrm{diag}(5, 5, 5)}$ $\boxed{\text{Justify}}$

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Name KEY

Differential Equations and Linear Algebra 2250-2

Final Exam 10:30am 13 Dec 2004

Ch7. (Linear Systems of Differential Equations)

☐ [40%] Ch7(a): Solve for the general solution $x(t)$, $y(t)$ in the system below.

$$\begin{aligned} x' &= x + y, \\ y' &= -9x + y. \end{aligned}$$

☐ [60%] Ch7(b): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

☐ [40%] Ch7(c): Assume A is 2×2 and has eigenvalues $2 \pm 3i$. In the system $\mathbf{u}' = A\mathbf{u}$ where $\mathbf{u}(t)$ has components $x(t)$, $y(t)$, explain why

$$x(t) = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t.$$

(a) $r^2 - 2r + 10 = 0$

$(r-1)^2 + 9 = 0$

$r = 1 \pm 3i$

$x = \boxed{e^{t}(c_1 \cos 3t + c_2 \sin 3t)}$

$y = x' - x$

$= \boxed{-3e^{t} c_1 \sin 3t + 3e^{t} c_2 \cos 3t}$

(b) $(-3-\lambda)((-3-\lambda)^2 - 1) = 0$

$(3+\lambda)(2+\lambda)(4+\lambda) = 0$

$\lambda = -2, -3, -4$

$\vec{x} = c_1 e^{-4t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

(c) we solve $\det(A - \lambda I) = 0$ to get $\lambda = 2 \pm 3i$, Then by The recipe

$x(t) = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t.$

we know A is not triangular, because Then A has a real eigenvalue.

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Name KEY **2250-2**

Differential Equations and Linear Algebra 2250-2

Final Exam 10:30am 13 Dec 2004

Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [30%] Ch10(a): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{s+3}{(s-1)(s+2)(s+1)^2}.$$

☐ [30%] Ch10(b): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. Do not solve for $x(t)$! Document steps by reference to tables and rules.

$$x''' + x' = te^{-t} + \cos\sqrt{2}\,t, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1.$$

☐ [35%] Ch10(c): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 2x + 3y, \\ y'' &= 3x + 5y, \\ x(0) &= 0, \quad x'(0) = 1, \\ y(0) &= 1, \quad y'(0) = 0. \end{aligned}$$

☐ [35%] Ch10(d): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\frac{s+1}{(s^2+2s+10)} + \frac{s+1}{s^2} + \frac{2+s}{s^2+4s}. \qquad \frac{2+s}{s(s+4)}$$

☐ [35%] Ch10(e): Find $\mathcal{L}(f(t))$, given $f(t) = \dfrac{e^{-3t} - e^{4t}}{t}$.

(a) $f = ae^{t} + be^{-2t} + cte^{-t}$ $\quad a = \frac{1}{3},\ b = -\frac{1}{3},\ c = -1$

(b) $\mathcal{L}(x) = \dfrac{1 + \frac{1}{(s+1)^2} + \frac{s}{s^2+2}}{s^3+s} =$

(c) $s^2\mathcal{L}(x) = 1 + 2\mathcal{L}(x) + 3\mathcal{L}(y)$
$s^2\mathcal{L}(y) = s + 3\mathcal{L}(x) + 5\mathcal{L}(y)$ $\quad \mathcal{L}(y) = \dfrac{\begin{vmatrix} s^2-2 & 1 \\ -3 & s \end{vmatrix}}{\begin{vmatrix} s^2-2 & -3 \\ -3 & s^2-5 \end{vmatrix}} =$

(d) $x = (-t)e^{-t}\cos 3t + 1 + t + \frac{1}{2} + \frac{1}{2}e^{-4t}$

(e) $\ln\left(\frac{s-4}{s+3}\right)$

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