

Differential Equations and Linear Algebra 2250-1
Midterm Exam 3, Spring 2005

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations.

1. (ch4) (a) State the subspace criterion [10%].

(b) Let V be the set of all vectors $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x + z = 3y$. Apply (a) to prove that V is a

subspace of \mathcal{R}^3 [40%].

(c) Find a 3×3 system of linear equations for the constants a, b, c in the partial fractions decomposition below [25%]. Solve for a, b, c by RREF methods [25%].

$$\frac{x^2 + 2x + 2}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2}$$

Ⓐ V is a subspace of vector space $S \iff \begin{cases} \text{(a) } \vec{0} \text{ is in } V \\ \text{(b) } \vec{v}_1, \vec{v}_2 \text{ in } V \implies c_1\vec{v}_1 + c_2\vec{v}_2 \text{ in } V \end{cases}$

Ⓑ Show (a): Zero is in V

Let $x=y=z=0$, then $x+z=0+0=0=3 \cdot 0=3y$, so $\vec{0}$ is in V .

Show (b): Combo is in V

Let $\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ be in V . Then $x_1+z_1=3y_1, x_2+z_2=3y_2$. Define

$x = c_1x_1 + c_2x_2, y = c_1y_1 + c_2y_2, z = c_1z_1 + c_2z_2$. Then

$$\begin{aligned} x+z &= c_1(x_1+z_1) + c_2(x_2+z_2) \\ &= c_1(3y_1) + c_2(3y_2) \\ &= 3(c_1y_1 + c_2y_2) \end{aligned}$$

Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = 3y$ is in V . The subspace criterion completes the proof.

Ⓒ $x^2 + 2x + 2 = a(x-1)(x+2) + b(x+2) + c(x-1)^2$

Subst $x=1, x=-2, x=0$. Then

$$\begin{cases} 5 = 3b \\ 2 = 9c \\ 2 = -2a + 2b + c \end{cases}$$

Then $b = 5/3, c = 2/9, a = 7/9$

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2. (ch5)

(a) Use the *recipe* for higher order constant-coefficient differential equations to write out the general solution of a higher order linear differential equation with characteristic equation $r^3(r^2-1)(r^2+4)^2(r-1)^2 = 0$ [50%].

(b) Given $mx''(t) + cx'(t) + kx(t) = 0$ and damped spring-mass system constants $m = 2$, $c = 4$, $k = 1$, solve the differential equation [40%] and classify the answer as over-damped, critically damped or under-damped [10%].

(a) roots = $0, 0, 0, 1, 1, 1, -1, 2i, -2i, 2i, -2i$

$$y = C_1 e^{0x} + C_2 e^x + C_3 e^{-x} + C_4 \cos 2x + C_5 \sin 2x$$

$$\left\{ \begin{array}{l} C_1 = c_1 + c_2 x + c_3 x^2 \\ C_2 = c_4 + c_5 x + c_6 x^2 \\ C_3 = c_7 \\ C_4 = c_8 + c_9 x \\ C_5 = c_{10} + c_{11} x \end{array} \right.$$

(b) $2r^2 + 4r + 1 = 0$

$$r = -1 + \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}$$

2 distinct real roots

over damped

$$x(t) = c_1 e^{-t+t/\sqrt{2}} + c_2 e^{-t-t/\sqrt{2}}$$

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3. (ch5)

(a) Determine for $y'' + y''' = x^2 \sin x + 1.55 + 5x^2 + \cos x$ the final form of a trial solution for y_p according to the method of undetermined coefficients [50%]. **Do not** find the system for the undetermined coefficients and don't solve for the coefficients!

(b) Apply variation of parameters to write an integral formula for a particular solution $x_p(t)$ for the equation $5x'' + 25x' + 20x = te^{-t}$ [50%]. **Don't integrate!**

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Ⓐ $r^5 + r^3 = 0$
 $r^3(r^2 + 1) = 0$
 roots = $0, 0, 0, i, -i$
 y_h -atoms = $1, x, x^2, \cos x, \sin x$

The atoms of RHS = $1, x, x^2, \cos x, x \cos x, x^2 \cos x, \sin x, x \sin x, x^2 \sin x$

$y = y_1 + y_2,$
 $y_1 = d_1 + d_2 x + d_3 x^2$
 $y_2 = (d_4 + d_5 x + d_6 x^2) \cos x + (d_7 + d_8 x + d_9 x^2) \sin x$

Fixup rule

$y_1 = x^3 (d_1 + d_2 x + d_3 x^2)$
 $y_2 = x (d_4 + d_5 x + d_6 x^2) \cos x + x (d_7 + d_8 x + d_9 x^2) \sin x$

Corrected trial sol $y = y_1 + y_2$

Ⓑ $x_p = \left(\int y_2 \left(\frac{-f}{W} \right) \right) y_1 + \left(\int y_1 \left(\frac{f}{W} \right) \right) y_2$

$x_h = \left(\int e^{-t} \left(\frac{-f}{W} \right) \right) e^{-4t} + \left(\int e^{-4t} \left(\frac{f}{W} \right) \right) e^{-t}$
 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -3e^{-5t}, \quad f = \frac{te^{-t}}{5}$

$x'' + 5x' + 4x = \frac{te^{-t}}{5}$

$r^2 + 5r + 4 = 0$

$(r+4)(r+1) = 0$

$y_1 = e^{-4t}$
 $y_2 = e^{-t}$

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4. (ch6)

Give an example of a 3×3 matrix A with eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and eigenvalues $-2, 5, -1$. Justify your answer.

use $AP = PD$, $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$P^{-1} = \frac{1}{|P|} \begin{pmatrix} -2 & +2 & -4 \\ 0 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix}^T$$

$$= \frac{-1}{2} \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{pmatrix}$$

 $|P| = -2$ by cofactor rule

$$A = PDP^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 10 & -1 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ -24 & 10 & 12 \\ -4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 0 & 0 \\ -12 & 5 & 6 \\ -2 & 0 & -1 \end{pmatrix}$$

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 70% if $AP = PD$ appear
 90% if $AP = PD, P, D$ appear.

5. (ch6)

Apply eigenanalysis [75%] to obtain the solution $\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + c_3 \mathbf{v}_3 e^{\lambda_3 t} + c_4 \mathbf{v}_4 e^{\lambda_4 t}$ of the system $\mathbf{x}' = A\mathbf{x}$ [25%], given

$$A = \begin{bmatrix} 4 & 1 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

$$0 = \begin{vmatrix} 4-\lambda & 1 & 1 & 0 \\ 1 & 4-\lambda & 1 & 0 \\ 0 & 0 & 4-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)(4-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda)(3-\lambda)(5-\lambda)$$

$$\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 4, \lambda_4 = 5$$

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Eigen vectors

$$\begin{pmatrix} 4 & 1 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \approx \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x=0$
 $y=0$
 $z=0$
 $w=t_1$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \approx \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = -t_1$
 $y = t_1$
 $z = 0$
 $w = 0$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = -t_1$
 $y = -t_1$
 $z = t_1$
 $w = 0$

$$\vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix} \approx \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = t_1$
 $y = t_1$
 $z = 0$
 $w = 0$

$$\vec{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = c_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} e^{4t} + c_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{5t}$$

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100% for 2 correct eigenpairs + $\vec{x}(t)$ -formula.
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