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# Differential Equations and Linear Algebra 2250-2

Midterm Exam 3, Spring 2005

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations.

**1. (ch4)** (a) State the **subspace criterion** [10%].

(b) Let $V$ be the set of all vectors $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $z = x + 2y$. Prove that $V$ is a subspace of $\mathcal{R}^3$ [40%].

(c) Find a $3 \times 3$ system of linear equations for the constants $a$, $b$, $c$ in the partial fractions decomposition below [25%]. Solve for $a$, $b$, $c$ by RREF methods [25%].

$$\frac{x^2 + x - 3}{(x+1)^2(x+2)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x+2}$$

(a) $V$ is a subspace of vector space $S$ $\iff \begin{cases} \text{(a) } \vec{0} \text{ in } V \\ \text{(b) } \vec{v}_1, \vec{v}_2 \text{ in } V \Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 \text{ in } V. \end{cases}$

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(b) Show $\vec{0}$ in $V$: Let $x=y=z=0$, Then $x+2y = 0+0 = 0 = z$ so $\vec{0}$ in $V$.

Show Comb in $V$: Let $x_1+2y_1 = z_1$ and $x_2+2y_2=z_2$. Then

$$c_1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + c_2\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1(x_1+2y_1) + c_2(x_2+2y_2) \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+2y \end{pmatrix} \quad \text{where } \begin{cases} x = c_1x_1 + c_2x_2 \\ y = c_1y_1 + c_2y_2 \end{cases}$$

So combinations are in $V$. The subspace criterion is satisfied.

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(c) $x^2+x-3 = a(x+1)(x+2) + b(x+2) + c(x+1)^2$

Choose $x=-1$, $x=-2$, $x=0$ to get

$$\begin{cases} -3 = 0 + b + 0 \\ -1 = 0 + 0 + c \\ -3 = 2a + 2b + c \end{cases}$$

Then $b=-3$, $c=-1$ and by back-subst, $2a = -3 - c - 2b$
$= -3 + 1 + 6$
$= 4$

$$\boxed{a=2,\ b=-3,\ c=-1}$$

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**2. (ch5)**
(a) Use the *recipe* for higher order constant-coefficient differential equations to write out the general solution of a higher order linear differential equation with characteristic equation $r^3(r^2-4)(r^2+4)^2(r+2)^2 = 0$ [50%].
(b) Given $mx''(t) + cx'(t) + kx(t) = 0$ and damped spring-mass system constants $m = 4$, $c = 3$, $k = 1$, solve the differential equation [40%] and classify the answer as over-damped, critically damped or under-damped [10%].

(a) roots $= 0, 0, 0, -2, -2, -2, 2, 2i, -2i, 2i, -2i$

$$y = \mathbb{C}_1 e^{0x} + \mathbb{C}_2 e^{-2x} + \mathbb{C}_3 e^{2x} + \mathbb{C}_4 \cos 2x + \mathbb{C}_5 \sin 2x$$

where
$$\begin{cases} \mathbb{C}_1 = c_1 + c_2 x + c_3 x^2 \\ \mathbb{C}_2 = c_4 + c_5 x + c_6 x^2 \\ \mathbb{C}_3 = c_7 \\ \mathbb{C}_4 = c_8 + c_9 x \\ \mathbb{C}_5 = c_{10} + c_{11} x \end{cases}$$

(b) $4r^2 + 3r + 1 = 0$

$$r = \frac{-3 \pm \sqrt{9-16}}{8}$$
$$= -\frac{3}{8} \pm \frac{\sqrt{7}}{8} i$$

$$\boxed{x(t) = \left[c_1 \cos\left(\frac{\sqrt{7}}{8}t\right) + c_2 \sin\left(\frac{\sqrt{7}}{8}t\right)\right] e^{-3t/8}}$$

$\boxed{\text{under-damped}}$

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**3. (ch5)**

(a) Determine for $y^v + 4y''' = x^2\cos 2x + 6.5x + 3x^2 + \sin 2x$ the final form of a trial solution for $y_p$ according to the method of undetermined coefficients [50%]. **Do not** find the system for the undetermined coefficients and don't solve for the coefficients!

(b) Apply variation of parameters to write an integral formula for a particular solution $x_p(t)$ for the equation $2x'' + 6x' + 4x = te^{-2t}$ [50%]. **Don't integrate!**

(a) $r^5 + 4r^3 = 0$

$r^3(r^2+4) = 0$

$r = 0, 0, 0, 2i, -2i$

atoms of $y_h$ = $1, x, x^2, \cos 2x, \sin 2x$

atoms for $y$ = $1, x, x^2, \cos 2x, x\cos 2x, x^2\cos 2x, \sin 2x, x\sin 2x, x^2\sin 2x$

Fixup rule: mult by powers of x to remove dups.

$$\boxed{\begin{aligned} y &= y_1 + y_2 \\ y_1 &= x^3(d_1 + d_2x + d_3x^2) \\ y_2 &= x(d_4 + d_5x + d_6x^2)\cos 2x \\ &\quad + x(d_7 + d_8x + d_9x^2)\sin 2x \end{aligned}}$$

(b) $x'' + 3x' + 2x = \frac{t}{2}e^{-2t}$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$W = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}$

$x_h(t) = c_1e^{-t} + c_2e^{-2t}$

$x_p(t) = \left(\int e^{-2t}\left(\frac{-f}{W}\right)\right)e^{-t} + \left(\int e^{-t}\frac{f}{W}\right)e^{-2t}$

$$\boxed{x_p(t) = \left(\int e^{-2t}\left(\frac{te^{-2t}}{2e^{-3t}}\right)dt\right)e^{-t} + \left(\int -e^{-t}\left(\frac{te^{-2t}}{2e^{-3t}}\right)dt\right)e^{-2t}}$$

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**4. (ch6)**
Give an example of a $3 \times 3$ matrix $A$ with eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and eigenvalues $-3$, $4$, $-1$ [60%]. Justify your answer [40%].

$$AP = PD \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 3 & 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$PD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 & 0 \\ 0 & 16 & -1 \\ -9 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det(P)}\begin{pmatrix} -4 & +3 & -12 \\ 0 & -1 & 0 \\ 0 & -1 & 4 \end{pmatrix}^T \qquad \det(P) = -4$$

$$= \frac{-1}{4}\begin{pmatrix} -4 & 0 & 0 \\ 3 & -1 & -1 \\ -12 & 0 & 4 \end{pmatrix}$$

$$A = PDP^{-1}$$

$$\boxed{A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 16 & -1 \\ -9 & 0 & 1 \end{pmatrix}\begin{pmatrix} -4 & 0 & 0 \\ 3 & -1 & -1 \\ -12 & 0 & 4 \end{pmatrix} \cdot \frac{-1}{4}}$$

$$\boxed{A = \begin{pmatrix} -3 & 0 & 0 \\ -15 & 4 & 5 \\ -6 & 0 & -1 \end{pmatrix}}$$

Grading: 50% for blank page or wrong idea.
70% if $AP=PP$ appears on the page
90% if $AP=PD$, $P$, $D$ appear.

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**5. (ch6)**
Apply eigenanalysis [75%] to obtain the solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} + c_4\mathbf{v}_4 e^{\lambda_4 t}$ of the system $\mathbf{x}' = A\mathbf{x}$ [25%], given

$$A = \begin{bmatrix} 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Grading: 10% for $\vec{x}(t)$
90% for eigenpairs
100% credit if 2 pairs are correct. Sign errors forgiven.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} (0-\lambda)(1-\lambda)$$
$$= ((1-\lambda)^2 - 4)(-\lambda)(1-\lambda)$$
$$= (1-\lambda-2)(1-\lambda+2)(-\lambda)(1-\lambda)$$
$$= (-1-\lambda)(3-\lambda)(-\lambda)(1-\lambda)$$

$$\boxed{\lambda = -1, 3, 0, 1}$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 2 & 4 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & -7 & -1 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 & 27 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = -2t_1$
$y = t_1$
$z = 0$
$w = 0$

$$\boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

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$\underline{\lambda = 3}$

$$\begin{pmatrix} -2 & 4 & 1 & 1 \\ 1 & -2 & 4 & 1 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -2 \end{pmatrix} \cong \begin{pmatrix} -2 & 4 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} -2 & 4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = 2t_1$
$y = t_1$
$z = 0$
$w = 0$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

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Similarly, $\vec{v}_3 = \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_4 = \begin{pmatrix} -17 \\ -5/4 \\ 4 \\ 1 \end{pmatrix}$. Then

$$\boxed{x = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} -17 \\ -5/4 \\ 4 \\ 1 \end{pmatrix} e^{t}}$$