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1. (rref)
Determine a, b such that (1) the system has no solution and (2) the system has a unique solution.

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 10y + 8z &= 3 \\ 3x + ay + bz &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 10 & 8 & 3 \\ 3 & a & b & 2 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & a-6 & b-3 & -1 \end{array}\right) \begin{array}{l} \text{combo} \\ \text{combo} \end{array}$$

$$\cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2/3 \\ 0 & 1 & 1 & 1/6 \\ 0 & a-6 & b-3 & -1 \end{array}\right)$$

$$\cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2/3 \\ 0 & 1 & 1 & 1/6 \\ 0 & 0 & b-3+6-a & -1+\frac{6-a}{6} \end{array}\right)$$

(1) No solution $b+3-a=0$, $-\frac{a}{6} \neq 0$

Signal eq "0=1"

(2) Unique solution $b+3-a \neq 0$
Three lead variables.

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2. **(vector spaces)**

(a) [25%] Let V be the vector space of functions $f(t) = c_1 + c_2 e^{2t} + c_3 e^{4t} + c_4(e^{2t} - e^{4t})$, for all values of c_1, c_2, c_3, c_4. Report a basis for V.

(b) [25%] Prove that the set S of all vectors $\mathbf{v}$ in $\mathcal{R}^3$ with $v_1 = 0$ is a subspace.

(c) [50%] Find a basis for the subspace of $\mathcal{R}^3$ given by the system of equations

$$\begin{aligned} x + 3y - 2z &= 0, \\ y + z &= 0, \\ x + 4y - z &= 0, \end{aligned}$$

(a) partials $= 1, e^{2t}, e^{4t}, e^{2t} - e^{4t}$

Basis $= 1, e^{2t}, e^{4t}$ because $e^{2t} - e^{4t}$ is a linear combo of basis elements listed.

(b) The vectors $\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}$ contain $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by taking $x_2 = x_3 = 0$.

Given $\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix}$ Then $c_1 \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 x_2 + c_2 y_2 \\ c_1 x_3 + c_2 y_3 \end{pmatrix}$ belongs to S. proof is complete, by the subspace criterion.

(c) $\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 1 & 4 & -1 \end{pmatrix} \cong \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\cong \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\cong \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x = 5t_1$

$y = -t_1$

$z = t_1$

Basis $= \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

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3. (independence)
Extract from the list below a largest set of independent vectors.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -3 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 0 \\ -3 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 6 \\ -3 \\ 0 \\ -2 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & -3 & 3 & 4 & 6 \\ -1 & 3 & -3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -3 & 0 & -2 \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & -3 & 3 & 4 & 6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix}$$

$$\cong \begin{pmatrix} \boxed{1} & -3 & 3 & 4 & 6 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot cols = 1, 4. The pivot cols of A are indep.

$\boxed{\vec{a}, \vec{d} \text{ are indep set, largest possible}}$

$$\boxed{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}}$$

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4. (determinants and elementary matrices)
Assume given 2×2 matrices A, B. Suppose $B = E_1E_2A$ and E_1, E_2 are 2×2 elementary matrices representing a combo rule and a swap rule. Explain precisely why $\det(2BA) = -4(\det(A))^2$.

$$
\begin{aligned}
\det(2BA) &= \det((2I)\cdot B\cdot A) \\
&= \det(2I)\det(B)\det(A) \\
&= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \det(B)\det(A) \\
&= 4\det(B)\det(A) \\
&= 4\det(E_1E_2A)\det(A) \\
&= 4\det E_1 \det E_2 (\det(A))^2 \\
&= 4(1)(-1)(\det(A))^2 \\
&= -4(\det(A))^2
\end{aligned}
$$

swap $\det E_2 = -1$
combo $\det E_1 = 1$

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5. (inverses and Cramer's rule)

Solve for just x_3 in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule: $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

$$x_3 = \frac{\Delta_3}{\Delta}$$

$$\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

cofactor exp

$$= 2\begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 2 & -2 & 1 \end{vmatrix}$$

cofactor exp

$$= 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -8$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

cof exp

$$= 2\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 2 & -2 & 0 \end{vmatrix}$$

cof exp

$$= 2\begin{vmatrix} 2 & 0 \\ 2 & -2 \end{vmatrix}$$

$$= -8$$

$$\boxed{x_3 = \frac{\Delta_3}{\Delta} = \frac{-8}{-8} = 1}$$

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