Differential Equations and Linear Algebra 2250

Sample Midterm Exam 3, Fall 2005

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

1. **(ch4)** Let A be a 51×51 matrix. Assume V is the set of all vectors $\mathbf{x}$ such that $A^2\mathbf{x} = 3\mathbf{x}$. Prove that V is a subspace of $\mathcal{R}^{51}$.

Solution 1. Use the subspace criterion: (a) Given $\mathbf{x}$ and $\mathbf{y}$ in V, write details to show $\mathbf{x}+\mathbf{y}$ is in V; (b) Given $\mathbf{x}$ in V and k constant, write details to show $k\mathbf{x}$ is in V. Details for (a): Given $A^2\mathbf{x} = 3\mathbf{x}$ and $A^2\mathbf{y} = 3\mathbf{y}$, add the equations to obtain the equation $A^2(\mathbf{x}+\mathbf{y}) = 3(\mathbf{x}+\mathbf{y})$. This finishes (a). Details for (b): Given $A^2\mathbf{x} = 3\mathbf{x}$ and k constant, multiply the equation by k and re-arrange factors to obtain the new equation $A^2(k\mathbf{x}) = 3(k\mathbf{x})$. This proves (b).

2. **(ch4)** Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [25%]. Solve for a, b, c, d, showing all **RREF** steps [60%]. Report the answers [15%].

$$\frac{x^2+2x-1}{(x+1)^2(x^2+6x+10)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c(x+3)+d}{x^2+6x+10}$$

Solution 2. Clear the fractions to get

$$x^2+2x-1 = a(x+1)(x^2+6x+10) + b(x^2+6x+10) + (c(x+3)+d)(x+1)^2.$$

Set $x = -1$ to get one equation for the constants. Choose 3 other values for x to obtain three other equations. Display the system of equations. Solve the system with RREF methods. The answer:

$$-\frac{2/5}{(x+1)^2} + \frac{8/25}{x+1} - \frac{(8/25)(x+3) - 19/25}{x^2+6\,x+10}.$$

3. **(ch5)** Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions:

1.[25%] $y'' + y' + y = 0$,

2.[25%] $y^{iv} + 4y'' = 0$,

3.[25%] Char. eq. $(r+2)^3(r^2-4)(r^2+4) = 0$,

4.[25%] Char. eq. $(r^2-3)^2(r^2+16)^3 = 0$.

Solution 3.
1: $r^2+r+1=0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{-x/2}\cos(\sqrt{3}x/2)$, $y_2 = e^{-x/2}\sin(\sqrt{3}x/2)$.
2: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = (c_1 + c_2x)e^{0x} + c_3\cos 2x + c_4 \sin 2x$.
3: Write as $(r+2)^4(r-2)(r^2+4) = 0$, then $y = u_1e^{-2x} + u_2e^{2x} + u_3\cos 2x + u_4 \sin 2x$. The polynomials are $u_1 = c_1 + c_2x + c_3x^2 + c_4x^3$ (4 terms for multiplicity 4), $u_2 = c_5$, $u_3 = c_6$, $u4 = c_7$.
4: Write as $(r-a)^2(r+a)^2(r^2+16)^3 = 0$ where $a = \sqrt{3}$. Then $y = u_1e^{ax} + u_2e^{-ax} + u_3\cos 4x + u_5 \sin 3x$. The polynomials are $u_1 = c_1 + c_2x$, $u_2 = c_3 + c_4x$, $u_3 = c_5 + c_6x + c_7x^2$, $u_4 = c_8 + c_9x + c_{10}x^2$.

4. **(ch5)** Given $4x''(t) + 4x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 4$, $k = 1$, solve the differential equation [70%] and classify the answer as over-damped, critically damped or under-damped [15%]. Illustrate in a physical model drawing the meaning of constants m, c, k [15%].

Solution 4.
Use $4r^2 + 4r + 1 = 0$ and the quadratic formula to obtain roots $r = -1/2, -1/2$. Case 2 of the recipe gives $y = (c_1 + c_2t)e^{-t/2}$. This is critically damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

5. **(ch5)** Determine for $y^{iv} - 9y'' = xe^{3x} + x^3 + e^{-3x} + \sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Solution 5.

The homogeneous solution is $y_h = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{-3x}$, because the characteristic polynomial has roots 0, 0, 3, -3.

1 An initial trial solution y is constructed for atoms 1, x, e^{3x}, e^{-3x}, $\cos x$, $\sin x$ giving

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= (d_1 + d_2 x)e^{3x}, \\ y_2 &= d_3 + d_4 x + d_5 x^2 + d_6 x^3, \\ y_3 &= d_7 e^{-3x}, \\ y_4 &= d_8 \cos x + d_9 \sin x. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation.

2 The fixup rule is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3, \\ y_1 &= x(d_1 + d_2 x)e^{3x}, \\ y_2 &= x^2(d_3 + d_4 x + d_5 x^2 + d_6 x^3), \\ y_3 &= x(d_7 e^{-3x}), \\ y_4 &= d_8 \cos x + d_9 \sin x. \end{aligned}$$

The powers of x multiplied in each case are designed to eliminate terms in the initial trial solution which duplicate atoms appearing in the homogeneous solution y_h. The factor is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the related atom in the homogeneous solution y_h. By design, unrelated atoms are unaffected by the fixup rule, and that is why y_4 was unaltered.

3 Undetermined coefficient step skipped, according to the problem statement.

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6. **(ch5)** Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 6x = 5\cos(3t)$.

Solution 6.

Solve $x'' + 2x' + 6x = 0$ by the recipe to get $x_h = c_1 x_1 + c_2 x_2$, $x_1 = e^{-t}\cos\sqrt{5}t$, $x_2 = e^{-t}\sin\sqrt{5}t$. Compute the Wronskian $W = x_1 x_2' - x_1' x_2 = \sqrt{5}e^{-2t}$. Then for $f(t) = 5\cos(3t)$,

$$x_p = x_1 \int x_2 \frac{-f}{W} dt + x_2 \int x_1 \frac{f}{W} dt.$$

The integrations are horribly difficult, so the method of choice is undetermined coefficients.

The trial solution is $x = d_1 \cos 3t + d_2 \sin 3t$. Substitute the trial solution to obtain the answers $d_1 = -1/3$, $d_2 = 2/3$. The unique periodic solution x_{SS} is extracted from the general solution $x = x_h + x_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). If $x_p = d_1 \cos 3t + d_2 \sin 3t = (1/3)(-\cos 3t + 2\sin 3t)$, then

$$x_{SS} = \frac{-1}{3}\cos 3t + \frac{2}{3}\sin 3t.$$

7. **(ch6)** Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

Solution 7.

Subtract λ from the diagonal elements of A and expand the determinant $\det(A - \lambda I)$ to obtain the characteristic polynomial $(1-\lambda)(1-\lambda)(4-\lambda)(1-\lambda) = 0$. The eigenvalues are the roots: $\lambda = 1, 1, 1, 4$. Used here was the *cofactor rule* for determinants. Sarrus' rule does not apply for 4×4 determinants (an error) and the triangular rule likewise does not directly apply (another error).

8. **(ch6)** Given a 3×3 matrix A has eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right);\quad \left(1, \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}\right);\quad \left(0, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}\right),$$

(a) find an invertible matrix P and a diagonal matrix D such that $AP = PD$ and (b) display Fourier's model for the equation $\mathbf{y} = A\mathbf{x}$.

Solution 8.

Details (a): According to the theory of diagonalizable matrices, P is the matrix package of eigenvectors and D is the matrix package of eigenvalues. There are $3! = 6$ possible orderings to make these packages, hence 6 possible answers exist, all of which are correct. Only one answer is given here. Since the eigenanalysis is given in the statement, then

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -5 & -3 \end{pmatrix},\quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Details (b): Fourier's model says $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$ implies $\mathbf{y} = A\mathbf{x} = x_1\lambda_1\mathbf{v}_1 + x_2\lambda_2\mathbf{v}_2 + x_3\lambda_3\mathbf{v}_3$. Then the display is

$$\mathbf{x} = x_1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

implies

$$\mathbf{y} = A\mathbf{x} = 3x_1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}.$$

9. **(ch6)** Give an example of a 3×3 matrix C which has exactly one eigenpair

$$\left(2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

Solution 9.

A diagonal matrix has its eigenvalues down the diagonal and its eigenvectors are the columns of the identity. A diagonal matrix won't produce an example, because it always has three eigenpairs.

The best idea is to make C triangular:

$$C = \begin{pmatrix} 2 & a & b \\ 0 & 2 & c \\ 0 & 0 & 2 \end{pmatrix}.$$

The twos down the diagonal are required so that the characteristic polynomial $\det(A-\lambda I) = 0$ becomes $(2-\lambda)^3 = 0$, hence there is only one eigenvalue $\lambda = 2$. If we want an essentially unique eigenvector, then in the parametric solution of the system $(A-(2)I)\mathbf{v} = \mathbf{0}$ there must be just one free variable t_1 (usually, we use t_1, t_2, t_3, ... for the free variable assignments) and then $\partial_{t_1} = \mathbf{v}$ where $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is given in the problem. Then the matrix

$$C - (2)I = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

has to have rank 2 and nullity 1. This happens for example if $a = c = 1$ and $b = 0$. Finally, we report C as constructed and check the answer by an eigenanalysis of C ($x = t_1$, $y = 0$, $z = 0$ is the parametric solution). The answer:

$$C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

10. **(ch6)** The eigenanalysis method says that the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then
(1) [75%] Display eigenanalysis details for A.
(2) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Solution 10.
Answer (1): The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Answer (2): The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$