Differential Equations and Linear Algebra 2250

Sample Midterm Exam 3, Fall 2004

Calculators, books, notes and computers are not allowed. Answer checks are not expected or required. First drafts are expected, not complete presentations. The midterm exam has 5 problems, some with multiple parts, suitable for 50 minutes.

1. **(ch4)** Let A be a 71×71 matrix. Assume V is the set of all vectors $\mathbf{x}$ such that $A^2\mathbf{x} = 3\mathbf{x}$. Prove that V is a subspace of $\mathcal{R}^{71}$.

Solution 1. Use the subspace criterion: (a) Given $\mathbf{x}$ and $\mathbf{y}$ in V, show $\mathbf{x}+\mathbf{y}$ is in V; (b) Given $\mathbf{x}$ in V and k constant, show $k\mathbf{x}$ is in V. Details for (a): Given $A^2\mathbf{x} = 3\mathbf{x}$ and $A^2\mathbf{y} = 3\mathbf{y}$, show $A^2(\mathbf{x}+\mathbf{y}) = 3(\mathbf{x}+\mathbf{y})$. To prove (a), add the equations for $.x$ and $\mathbf{y}$. Details for (b): Given $A^2\mathbf{x} = 3\mathbf{x}$ and k constant, show $A^2(k\mathbf{x}) = 3(k\mathbf{x})$. To prove (b), multiply the equation for $\mathbf{x}$ by k and re-arrange factors.

2. **(ch4)** Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [25%]. Solve for a, b, c, d, showing all **RREF** steps [60%]. Report the answers [15%].

$$\frac{x^2+2x-1}{(x+1)^2(x^2+6x+10)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c(x+3)+d}{x^2+6x+10}$$

Solution 2. Clear the fractions to get

$$x^2+2x-1 = a(x+1)(x^2+6x+10) + b(x^2+6x+10) + (c(x+3)+d)(x+1)^2.$$

Set $x=-1$ to get one equation for the constants. Choose 3 other values for x to obtain three different equations. Write the system of equations and solve it with RREF methods. The answer:

$$-\frac{2/5}{(x+1)^2} + \frac{8/25}{x+1} - \frac{1}{25}\frac{5+8\,x}{x^2+6\,x+10}.$$

3. **(ch5)** Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions: **1**.[50%] $y'' + y' + y = 0$, **2**.[50%] $y^{iv} + 4y'' = 0$.

Solution 3.

1: $r^2+r+1=0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{x/2}\cos(\sqrt{3}x/2)$, $y_2 = e^{x/2}\sin(\sqrt{3}x/2)$.

2: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = (c_1+c_2x)e^{0x} + c_3\cos 2x + c_4 \sin 2x$.

4. **(ch5)** Given $4x''(t) + 4x'(t) + x(t) = 0$, which represents a damped spring-mass system with $m=4$, $c=4$, $k=1$, solve the differential equation [70%] and classify the answer as over-damped, critically damped or under-damped [15%]. Illustrate in a physical model the meaning of m, c, k [15%].

Solution 4.

Use $4r^2+4r+1=0$ and the quadratic formula to obtain roots $r=-1/2, -1/2$. Case 2 of the recipe gives $y = (c_1+c_2t)e^{-t/2}$. This is critically damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

5. **(ch5)** Determine for $y^{iv} - 9y'' = xe^{3x} + x^3 + e^{-3x}$ the final form of a trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Solution 5.

The homogeneous solution is $y_h = c_1 + c_2x + c_3e^{3x} + c_4e^{-3x}$.

An initial trial solution is constructed for atomic problems $y_1^{iv} - 9y_1'' = xe^{3x}$, $y_2^{iv} - 9y_2'' = x^3$, $y_3^{iv} - 9y_3'' = e^{-3x}$, giving

$$\begin{aligned} y_1 &= (d_1+d_2x)e^{3x}, \\ y_2 &= d_3 + d_4x + d_5x^2 + d_6x^3, \\ y_3 &= d_7e^{-3x}. \end{aligned}$$

The given linear combinations of independent functions are supposed to reproduce, by assignment of constants, all derivatives of the right side of the differential equation.

The fixup rule is applied individually to each of y_1, y_2, y_3 to give the **final trial solutions**

$$\begin{aligned} y_1 &= x(d_1 + d_2 x)e^{3x}, \\ y_2 &= x^2(d_3 + d_4 x + d_5 x^2 + d_6 x^3), \\ y_3 &= x(d_7 e^{-3x}). \end{aligned}$$

The powers of x multiplied in each case are designed to eliminate terms of duplication in the homogeneous solution y_h.

6. (ch5) Find by variation of parameters or undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 6x = 5\cos(3t)$.

Solution 6.
Solve $x'' + 2x' + 6x = 0$ by the recipe to get $x_h = c_1 x_1 + c_2 x_2$, $x_1 = e^{-t}\cos\sqrt{5}t$, $x_2 = e^{-t}\sin\sqrt{5}t$. Compute the Wronskian $W = x_1 x_2' - x_1' x_2 = \sqrt{5}e^{-2t}$. Then for $f(t) = 5\cos(3t)$,

$$x_p = x_1 \int x_2 \frac{-f}{W} dt + x_2 \int x_1 \frac{f}{W} dt.$$

The integrations are horribly difficult, so the method of choice is undetermined coefficients.

The trial solution is $x = A\cos 3t + B\sin 3t$. Substitute the trial solution to obtain the answers $A = -1/3$, $B = 2/3$. The unique periodic solution is then

$$x = \frac{-1}{3}\cos 3t + \frac{2}{3}\sin 3t.$$

7. (ch6) Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

Solution 7.
Subtract λ from the diagonal elements of A and expand the determinant $\det(A - \lambda I)$ by cofactors to obtain the characteristic polynomial $(1-\lambda)(1-\lambda)(4-\lambda)(1-\lambda = 0)$. The eigenvalues are the roots: $\lambda = 1, 1, 1, 4$.

8. (ch6) Given a 3×3 matrix A has eigenpairs

$$3, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}; \quad 1, \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}; \quad 0, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix},$$

find an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Solution 8.
According to the theory of diagonalizable matrices, P is the matrix of eigenvectors and D is the matrix of eigenvalues. Then

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -5 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

9. (ch6) Give an example of a 3×3 matrix C which has exactly one eigenpair

$$2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solution 9.
The best idea is to make C triangular:

$$C = \begin{pmatrix} 2 & a & b \\ 0 & 2 & c \\ 0 & 0 & 2 \end{pmatrix}.$$

The twos down the diagonal are required so that the characteristic polynomial $\det(A-\lambda I)=0$ becomes $(2-\lambda)^3=0$, hence there is only one eigenvalue $\lambda=2$. If we want an essentially unique eigenvector, then in the parametric solution of the system $(A-(2)I)\mathbf{v}=\mathbf{0}$ there must be just one free variable t_1 (usually, we use t_1, t_2, t_3, ... for the free variable assignments) and then $\partial_{t_1}=\mathbf{v}$ where $\mathbf{v}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ is given in the problem. Then the matrix

$$C-(2)I=\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}$$

has to have rank 2 and nullity 1. This happens for example if $a=c=1$ and $b=0$. Finally, we report C as constructed and check the answer by an eigenanalysis of C ($x=t_1$, $y=0$, $z=0$ is the parametric solution).

10. (ch7) Solve for $x(t)$, $y(t)$ in the system below. The answers depend upon two arbitrary constants, because $x(0)$ and $y(0)$ are not supplied.

$$\begin{aligned}x' &= x-y,\\ y' &= 10x+y.\end{aligned}$$

Solution 10.
Let $A=\begin{pmatrix}1&-1\\10&1\end{pmatrix}$. Then $\det(A-\lambda I)=0$ is the polynomial equation $\lambda^2-2\lambda+11=0$ with roots $1\pm i\sqrt{10}$. The second order recipe implies the first answer (for x) is

$$x(t)=c_1e^t\cos\sqrt{10}t+c_2e^t\sin\sqrt{10}t.$$

Solve the first equation for $y=x-x'$ and expand from the preceding formula for $x(t)$ to give the second answer

$$y(t)=\sqrt{10}c_1e^t\sin\sqrt{10}t-\sqrt{10}c_2e^t\cos\sqrt{10}t.$$

11. (ch7) Apply the eigenanalysis method to solve the system $\mathbf{x}'=A\mathbf{x}$, given $A=\begin{bmatrix}4&1&1\\1&4&1\\0&0&4\end{bmatrix}$.

Solution 11.
The eigenpairs are

$$5,\begin{pmatrix}1\\1\\0\end{pmatrix};\quad 4,\begin{pmatrix}-1\\-1\\1\end{pmatrix};\quad 3,\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

The method implies

$$\mathbf{x}(t)=c_1e^{5t}\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2e^{4}\begin{pmatrix}-1\\-1\\1\end{pmatrix}+c_3e^{3t}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$