

Math 2250
Maple Project 2 Part A: Linear Algebra
October 2005

Due date: See the internet due dates. Maple lab 2 has parts A (problems 2.1, 2.2, 2.3) and B (problems 2.4, 2.5, 2.6), issued in two different documents. This document is part A.

References: Code in maple appears in 2250mapleL2a-F2005.txt at URL <http://www.math.utah.edu/~gustafso/>. This document: 2250mapleL2a-F2005.pdf.

Problem 2.1. (Matrix Algebra)

Define $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$. Create a worksheet in maple which states this problem in text, then defines the four objects. The worksheet should contain text, maple code and displays. Continue with this worksheet to answer (1)–(7) below. Submit problem 2.1 as a worksheet print on 8.5 by 11 inch paper. See Example 1 for maple commands.

- (1) Compute AB and BA . Are they the same?
- (2) Compute $A + B$ and $B + A$. Are they the same?
- (3) Let $C = A + B$. Compare C^2 to $A^2 + 2AB + B^2$. Explain why they are different.
- (4) Compute the transpose of AB and compare it to the product of the transpose of A with the transpose of B , multiplied in the correct order so that you expect equality.
- (5) Solve for \mathbf{X} in $B\mathbf{X} = \mathbf{v}$ by three different methods.
- (6) Solve $A\mathbf{Y} = \mathbf{v}$ for \mathbf{Y} . Do an answer check.
- (7) Solve $A\mathbf{Z} = \mathbf{w}$. Explain your answer.

Problem 2.2. (Row space)

Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 1 & 2 & -3 & -1 & -9 \end{pmatrix}$. Find three different bases for the row space of A , using the following methods.

1. The method of Example 2, below.
2. The maple command `rowSPACE(A)`.
3. The `rref`-method: select rows from `rref(A)`.

Verify that all three bases are equivalent.

Problem 2.3. (Matrix Equations)

Let $A = \begin{pmatrix} 8 & 10 & 3 \\ -3 & -5 & -3 \\ -4 & -4 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Let P denote a 3×3 matrix. Assume the following result (proved below):

Lemma 1. The equality $AP = PT$ holds if and only if the columns \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 of P satisfy $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = -2\mathbf{v}_2$, $A\mathbf{v}_3 = 5\mathbf{v}_3$.

- (a) Determine three specific columns for P such that $\det(P) \neq 0$ and $AP = PT$. Infinitely many answers are possible. See Example 4 for the maple method that determines a column of P . (b) After reporting the three columns, check the answer by computing $AP - PT$ (it should be zero) and $\det(P)$ (it should be nonzero).

Example 1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$. Define and display matrix A and vector \mathbf{b} in **maple**. Then compute

- (1) The inverse of A .
- (2) The augmented matrix $C = \mathbf{aug}(A, \mathbf{b})$.
- (3) The reduced row echelon form $R = \mathbf{rref}(C)$.
- (4) The column \mathbf{X} of R which solves $A\mathbf{X} = \mathbf{b}$.
- (5) The matrix A^3 .
- (6) The transpose of A .
- (7) The matrix $AC - 3A^2$.
- (8) The solution \mathbf{X} of $A\mathbf{X} = \mathbf{b}$ by two methods different than (4).

Solution: To get help, enter ?linalg into a worksheet, then select commands that match ones below.

```
with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,-1]]);
b:=vector([9,8,3]);
print("(1)"); inverse(A);
print("(2)"); C:=augment(A,b);
print("(3)"); R:=rref(C);
print("(4)"); X:=col(R,4);
print("(5)"); evalm(A^3);
print("(6)"); transpose(A);
print("(7)"); evalm(A&*C-3*(A^2));
print("(8)"); X:=linsolve(A,b); X:=evalm(inverse(A) &* b);
```

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$.

- (1) Find a basis for the column space of A .
- (2) Find a basis for the row space of A .
- (3) Find a basis for the nullspace of A .
- (4) Find $\mathbf{rank}(A)$ and $\mathbf{nullity}(A)$.
- (5) Find the dimensions of the nullspace, row space and column space of A .

Solution: The theory applied: *The columns of B corresponding to the leading ones in $\mathbf{rref}(B)$ are independent and form a basis for the column space of B .* Results for the row space can be obtained by applying the above theory to the transpose of the matrix.

The **maple** code which applies is

```
with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
           [ 2, 3,-2, 1,-3],
           [ 3, 5,-5, 1,-8],
           [ 4, 3, 8, 2, 3]]);
print("(1)"); C:=rref(A); # leading ones on columns 1,2,4
BASIScolumnspace:=col(A,1),col(A,2),col(A,4);
print("(2)"); F:=rref(transpose(A)); # leading ones on columns 1,2,3
```

```

BASISrowSpace=row(A,1),row(A,2),row(A,3);
print("(3)"); nullspace(A); linsolve(A,vector([0,0,0,0]));
print("(4)"); RANK=rank(A); NULLITY=coldim(A)-rank(A);
print("(5)"); DIMnullspace=coldim(A)-rank(A); DIMrowSpace=rank(A);
DIMcolumnSpace=rank(A);

```

Example 3. Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & -2 & 1 & -3 \\ 3 & 5 & -5 & 1 & -8 \\ 4 & 3 & 8 & 2 & 3 \end{pmatrix}$. Verify that the following column space bases of A are equivalent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 17 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -9 \end{pmatrix}.$$

Solution: The maple code which applies is

```

with(linalg):
A:=matrix([[ 1, 1, 1, 2, 6],
            [ 2, 3,-2, 1,-3],
            [ 3, 5,-5, 1,-8],
            [ 4, 3, 8, 2, 3]]);
v1:=vector([1, 2, 3, 4]); v2:=vector([1, 3, 5, 3]); v3:=vector([2, 1, 1, 2]);
w1:=vector([1, 0, 0, -3]); w2:=vector([0, 1, 0, 17]); w3:=vector([0, 0, 1, -9]);
F:=augment(v1,v2,v3);
G:=augment(w1,w2,w3);
rank(A); rank(F); rank(G);
rank(augment(A,v1))-rank(A);
rank(augment(A,v2))-rank(A);
rank(augment(A,v3))-rank(A);
rank(augment(A,w1))-rank(A);
rank(augment(A,w2))-rank(A);
rank(augment(A,w3))-rank(A);

```

The theory says that two bases of a subspace S of \mathcal{R}^4 are equivalent. We justify that the proposed sets are independent and have the correct size ($\mathbf{rank}(F) = \mathbf{rank}(G) = \mathbf{rank}(A)$), and that all six vectors are in the column space of A ($\mathbf{rank}(\mathbf{aug}(A, \mathbf{v})) = \mathbf{rank}(A)$ if and only if $A\mathbf{X} = \mathbf{v}$ has a solution \mathbf{X}).

Example 4. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$. Solve the equation $A\mathbf{x} = -3\mathbf{x}$ for \mathbf{x} .

The maple details appear below. The idea is to write the problem as a homogeneous problem $(A + 3I)\mathbf{x} = \mathbf{0}$, which always has a solution.

```

with(linalg):
A:=matrix([[1,2,3],[2,-1,1],[3,0,0]]);
linsolve(evalm(A+3*diag(1,1,1)),vector([0,0,0]));
# ans: t_1*vector([-2,1,2])

```

Proof of Lemma 1. Define $r_1 = 1$, $r_2 = -2$, $r_3 = 5$. Assume $AP = PT$, $P = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $T = \mathbf{diag}(r_1, r_2, r_3)$. The definition of matrix multiplication implies that $AP = \mathbf{aug}(A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3)$ and $PT = \mathbf{aug}(r_1\mathbf{v}_1, r_2\mathbf{v}_2, r_3\mathbf{v}_3)$. Then $AP = PT$ holds if and only if the columns of the two matrices match, which is equivalent to the three equations $A\mathbf{v}_1 = r_1\mathbf{v}_1$, $A\mathbf{v}_2 = r_2\mathbf{v}_2$, $A\mathbf{v}_3 = r_3\mathbf{v}_3$. The proof is complete.

End of Maple Lab 2 Part A.