Applied Differential Equations 2250-2
Midterm Exam 1
Wednesday, 16 February 2005

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for \( y(x) \) in the equation \( y' = 2xe^{-2x} - \sec^2 x + \frac{x^3}{1+x^2} \).

\[
\int y' \, dx = \int \left( 2xe^{-2x} - \sec^2 x + \frac{x^3}{1+x^2} \right) \, dx
\]

\[
= \int 2xe^{-2x} \, dx - \int \sec^2 x \, dx + \int \frac{x^3}{1+x^2} \, dx
\]

\[
= -xe^{-2x} + \frac{e^{-2x}}{-2} - \tan x + \int \left( \frac{x^3}{1+x^2} + \frac{-x}{1+x^2} \right) \, dx
\]

\[
= -xe^{-2x} - \frac{e^{-2x}}{2} - \tan x + \int x \, dx - \int \frac{x}{1+x^2} \, dx
\]

\[
= (-x - \frac{1}{2})e^{-2x} - \tan x + \frac{x^2}{2} - \frac{1}{2} \ln (1+x^2) + C
\]

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Separable Equation Test)

The problem \( y' = 2x - x^{5/3} - 2xy^2 + x^{5/3}y^3 \) may or may not be separable. If it is, then write formulae for \( F, G \) and decompose the problem as \( y' = F(x)G(y) \). Otherwise, explain in detail why it fails to be separable. Do not solve for \( y! \)

\[
\begin{align*}
    f(x,y) & = 2x - x^{5/3} - 2xy^2 + x^{5/3}y^3 \\
    f(1,0) & = 2 - 1 \\
            & = 1
\end{align*}
\]

\[
\begin{align*}
    F(x) & = \frac{f(x,0)}{f(1,0)} \\
          & = 2x - x^{5/3}
\end{align*}
\]

\[
\begin{align*}
    G(y) & = \frac{f(1,y)}{f(1,0)} \\
          & = 1 - 2y^2 + y^3
\end{align*}
\]

\[
\begin{align*}
    FG & = (2x - x^{5/3})(1 - 2y^2 + y^3) \\
        & = 2x - x^{5/3} - 4xy^2 + 2x^{5/3}y^2 + 2xy^3 - x^{5/3}y^3 \\
        & = 2x - x^{5/3} - 2xy^2 + x^{5/3}y^3
\end{align*}
\]

\[
\begin{align*}
    FG & \neq f \quad \Rightarrow \quad \text{NOT separable.}
\end{align*}
\]

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3. (Solve a Separable Equation)

Given \( \frac{dy}{dx} = \frac{x^2 + 2x}{3 + 2x} (1 - 5y^2) \), find the non-equilibrium solution in implicit form. Do not solve for \( y \) explicitly and do not find equilibrium solutions.

\[
\int \frac{yy'}{1-5y^2} \, dx = \int \frac{x^2 + 2x}{3 + 2x} \, dx
\]

\[
- \frac{1}{10} \ln |1 - 5y^2| = \int \left( \frac{1}{2} x + \frac{1}{4} + \frac{-3/4}{3 + 2x} \right) \, dx
\]

\[
- \frac{1}{10} \ln |1 - 5y^2| = \frac{x^2}{4} + \frac{x}{4} - \frac{3}{4} \ln |3 + 2x| + C
\]

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4. (Linear Equations)

Solve (a) \( 3\nu'(t) = 15 - \frac{5}{t+11} \nu(t), \nu(0) = 2 \). Show all integrating factor steps.

(b) \( y'(t) = v(t), y(0) = 10 \). Show all quadrature steps.

\[ \begin{align*}
\text{(a)} \quad \nu' &= 5 - \frac{\alpha}{t+11} \nu, \quad \nu(0) = 2 \quad \left[ \alpha = \frac{5}{11} \right] \\
\nu' + \frac{\alpha}{t+11} \nu &= 5 \\
(\alpha \nu)' &= 5 \\
\alpha \nu &= 5 \int_0^t \alpha \, dt \\
&= \frac{5}{11} (t+11) + c \\
\nu &= \frac{5}{11} (t+11) + c (t+11)^{-\frac{11}{5}} \\
2 &= \frac{55}{11} + c \\
c &= \left( 2 - \frac{55}{11} \right) (t+11)^{-\frac{11}{5}} \\
\nu &= \frac{155}{8} - \frac{165}{8} (t+11) + \left( 2 - \frac{165}{8} \right) (t+11)^{-\frac{11}{5}} \\
\nu &= \frac{155}{8} + \frac{165}{8} + \left( 2 - \frac{165}{8} \right) (t+11)^{-\frac{11}{5}} \\
\end{align*} \]

\[ \begin{align*}
\text{(b)} \quad \int_0^t y' \, dt &= \int_0^t (\nu(0)) \, dt \\
\int_0^t y' \, dt &= \int_0^t \left( \frac{155}{8} + \frac{165}{8} + \left( 2 - \frac{165}{8} \right) \frac{11}{(t+11)} \right) \, dt \\
y &= 10 + \frac{155}{16} + \frac{165}{8} + \left( 2 - \frac{165}{8} \right) \frac{11}{(t+11)} \left( \frac{11}{t+11} \right)^{-\frac{2}{3}} \\
\end{align*} \]

Full credit for method. Answer not checked.

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5. (Stability)
(a) Draw a phase line diagram for the chemical reaction equation \( \frac{dx}{dt} = (2 - 3x)^5(1 - x)^2(2 - x)x^3 \). Expected in the diagram are equilibrium points, signs of \( x' \) and flow direction markers (< and >).
(b) Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, stable, unstable.

\( \text{Equilibrium points are } \frac{2}{3}, 1, 2, 0 \)

\[
\begin{align*}
\frac{f(x)}{f(-1)} &= (+)(+)(+)(-) = (-) \\
\frac{f(0.25)}{f(1.5)} &= (+)(+)(+)(+) = (+) \\
\frac{f(1.5)}{f(3)} &= (-)(+)(-)(+) = (+)
\end{align*}
\]

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