Ch3. (Linear Systems and Matrices)

[50%] Ch3(a): Find the second entry on the third row of the inverse matrix $B^{-1}$ by the formula $B^{-1} = \text{adj}(B)/\text{det}(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of $3 \times 3$ Sarrus' rule is disallowed ($2 \times 2$ use is allowed).

\[
B = \begin{bmatrix}
1 & 1 & -2 & 0 \\
1 & 1 & 0 & 0 \\
0 & -1 & 2 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

[50%] Ch3(b): Determine all values of $k$ such that the system $Rx = f$ has (1) infinitely many solutions [25%], (2) a unique solution [15%] and (3) no solution [10%].

\[
R = \begin{bmatrix}
2 & 1 & -2k \\
2 & 4k & -2 \\
0 & 0 & 4
\end{bmatrix}, \quad f = \begin{pmatrix}
0 \\
1 - 4k
\end{pmatrix}
\]

[25%] Ch3(c): Let $A$ be a $41 \times 42$ triangular matrix ($a_{ij} = 0$ for $i > j$) with coefficients either 0 or 1. Find infinitely many solutions $x$ for $Ax = 0$.

[25%] Ch3(d): An example exists of a square matrix $A$ such that $A^2$ is not the zero matrix but $A^3 = 0$. Prove that the example is not a diagonal matrix.

\( A = B^{-1} = \frac{\text{adj}(B)}{\text{det}(B)} \), $a_{32} = (-1)^{3+2} \text{cof}(B_{2,3})/\text{det}(B) = (-1) \begin{vmatrix}
1 & 0 \\
1 & 2
\end{vmatrix} = 1/2$

\( A = \text{adj}(R, f) = \begin{bmatrix} 2 & 1 & -2k \\ 2 & 4k & -2 \\ 0 & 0 & 4 \end{bmatrix} \)

\( 2 \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2k \\ -4k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) using combo & mult,

(1) many sols when free var exists $\iff x = 0 \iff k = 1/4$

(2) unique sol when zero free vars $\iff x \not= 0 \iff k \neq 1/4$

(3) no solution, never happens

\( A \) has rank $\leq 41$, there must be one free var, so many sols. To find some, set a free variable equal to 1 and all other free vars zero.

\( A = \text{diag}(a_1, \ldots, a_n), \quad A^n = \text{diag}(a_1^n, \ldots, a_n^n) \), so $A^2 = 0$ implies $A = 0$ and then $A^2 = 0$, a contradiction.

Staple this page to the top of all Ch3 work. Submit one package per chapter.
Ch4. (Vector Spaces)

[40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors \( v_1, v_2, v_3 \) in \( \mathbb{R}^3 \). Apply the test to the vectors below. Report independent or dependent.

\[
\begin{align*}
v_1 &= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}.
\end{align*}
\]

[60%] Ch4(b): Let \( A \) be a \( 23 \times 23 \) matrix. Define \( V \) to be the set of all vectors \( x \) in \( \mathbb{R}^{23} \) such that \( A(A-I)x + (I-A)(I+A)x + 2x = 0 \). Prove that \( V \) is a subspace of \( \mathbb{R}^{21} \).

[60%] Ch4(c): Find a basis of fixed vectors in \( \mathbb{R}^4 \) for (1) the column space of the \( 4 \times 4 \) matrix \( A \) below and (2) the row space of the \( 4 \times 4 \) matrix \( A \) below. The reported basis must consist of columns of \( A \) and rows of \( A \), respectively.

\[
A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 11 \\ 2 & 2 & -2 & 1 \end{pmatrix}
\]

[40%] Ch4(d): Find a \( 4 \times 4 \) system of linear equations for the constants \( a, b, c, d \) in the partial fractions decomposition below. Solve for \( a, b, c, d \), showing all RREF steps. Report the answers.

\[
\frac{x^2 - 3x + 1}{(x + 1)^2(x - 2)^2} = \frac{a}{x + 1} + \frac{b}{(x + 1)^2} + \frac{c}{x - 2} + \frac{d}{(x - 2)^2}
\]

\[\begin{align*}
\text{The vectors are independent} &\iff \text{rref \( \text{aug} (v_1, v_2, v_3) \) is \( I \).} \\
\begin{pmatrix} -1 & 3 & 4 \\ 2 & 1 & 4 \end{pmatrix} &\equiv \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} &\equiv \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} &\Rightarrow \text{dependent}
\end{align*}\]

\[\begin{align*}
\text{Let } B &= A(A-I) + (I-A)(I+A) + 2I, \quad \text{Then } V = \{ \vec{x} : B\vec{x} = \vec{0} \}.
\end{align*}\]

\[\begin{align*}
\text{already, } \vec{0} \text{ is in } V. \quad \text{By superposition, for } a,b,c,d \\
E_x (c_1 \vec{x}_1 + c_2 \vec{x}_2), \quad \text{it follows that } B\vec{x}_1 = B\vec{x}_2 = \vec{0} \Rightarrow B(c_1 \vec{x}_1 + c_2 \vec{x}_2) = \vec{0} \Rightarrow \text{Subspace criterion holds for } V.
\end{align*}\]

\[\begin{align*}
\text{rref}(A) &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{cols 1, 4} \big| A \text{ are independent, form a col space basis.} \\
\end{align*}\]

\[\begin{align*}
\text{Doing the same for } A^T: \text{rref}(A^T) &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rows 1, 2,}\end{align*}\]

Clear fractions, Substitute \( x = (1, 2, 0, 1) \), Solve \( \begin{align*}
a = -\frac{5}{27}, & b = \frac{5}{9}, c = \frac{5}{27}, d = -\frac{1}{9}.
\end{align*}\]

Staple this page to the top of all Ch4 work. Submit one package per chapter.
Ch5. (Linear Equations of Higher Order)

[30%] Ch5(a): Using the recipe for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

1. \[ (r^2 - 5r)^2(r^2 - 25) = 0. \]
2. \[ (r - 4)^2(r^2 + 2r + 5)^2(r^2 - 16)^3 = 0 \]

[30%] Ch5(b): Given a damped spring-mass system \( mx''(t) + cx'(t) + kx(t) = 0 \) with \( m = 10, \ c = 11, \ k = 3 \), solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

[40%] Ch5(c): Determine the final form of a trial solution for \( y_p \) according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

\[ y^{iv} + 18y'' + 81y = x^2e^{3x} + x \cos 3x + e^{-3x} + e^{3x} \sin 3x \]

[30%] Ch5(d): Find the steady-state periodic solution for the equation

\[ x'' + 2x' + 17x = \cos(4t). \]

\[ \text{Roots: } r = 0, 0, 5, 5, 5, -5; \quad y = c_1e^{5x} + c_2e^{-5x} \]

\[ \text{Roots: } \lambda = 4, 0, 0, 4, 4, -4, -4, -4, -4, -4, 1, 2, 1, 2, 1, 2, -2, -2 \]

\[ y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{4x} + (c_6 + c_7 x + c_8 x^2) e^{-4x} \]

\[ + (c_9 + c_{10} x) e^x \cos(2x) + (c_{11} + c_{12} x) e^x \sin(2x) \]

\[ 10r^2 + 11r + 3 = (2r+1)(5r+3) = 0; \quad x(t) = c_1e^{-t/2} + c_2e^{-3t/5}; \quad \text{Overdamped} \]

\[ r^4 + 18r^2 + 81 = (r^2 + 9)^2; \quad y_p \text{ has atoms } x \cos 3x, \sin 3x, x \cos 3x, \sin 3x \]

\[ y_{\text{trial}} = \text{combo of atoms } x^2e^{3x}, xe^{3x}, e^{3x}, e^{-3x}, x \cos 3x, x \sin 3x, \]

\[ = 10 \text{ atoms } x \cos 3x, \sin 3x, e^{3x} \cos 3x, e^{3x} \sin 3x. \]

\[ \text{Fixup: multi atoms } x \cos 3x, x \sin 3x, x \cos 3x, \sin 3x. \]

\[ y = \text{Terms of } y_{\text{trial}}, \text{ less related terms } + \text{ combo of } x^2 \cos 3x, x^2 \sin 3x, x \cos 3x, x \sin 3x. \]

\[ x(t) = \frac{1}{65} \cos(4t) + \frac{2}{65} \sin(4t), \text{ use standard undetermined coeff.} \]

Staple this page to the top of all Ch5 work. Submit one package per chapter.
Ch6. (Eigenvalues and Eigenvectors)

[30%] Ch6(a): Find the eigenvalues of the matrix $A$:

$$
A = \begin{bmatrix}
3 & 1 & -1 & 0 \\
0 & 2 & -2 & 1 \\
0 & 0 & 4 & 1 \\
0 & 0 & 1 & 4 \\
\end{bmatrix}
$$

[35%] Ch6(b): Let $A$ be a $2 \times 2$ matrix with eigenpairs $(1, \mathbf{v}_1), (2, \mathbf{v}_2)$.

Display formulae for two distinct diagonal matrices $D_1, D_2$ and two invertible $2 \times 2$ matrices $P_1, P_2$ such that $AP_1 = P_1 D_1$ and $AP_2 = P_2 D_2$ [15%]. Justify your claims [20%].

[35%] Ch6(c): Assume $\det(A - \lambda I) = \det(B - \lambda I)$ for two $5 \times 5$ matrices $A, B$. Let $A$ have eigenvalues $1, 2, 3, 4 \pm 5i$. Find the eigenvalues of $(5/4)B + I$, where $I$ is the identity matrix.

[35%] Ch6(d): An example exists of a $3 \times 3$ matrix $C$ which has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ but only two independent eigenvectors. Display an example $C$ where $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Justify by doing the eigenanalysis of $C$.

\[\begin{align*}
\text{a) } & \lambda = 2, 3, 3, 5 \quad \text{by cofactor expansion} \\
\text{b) } & D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1 = \text{span}(\mathbf{v}_1, \mathbf{v}_2), \quad P_2 = \text{span}(\mathbf{v}_2, \mathbf{v}_1). \quad \text{Valid by } AP = PD. \\
\text{c) } & \det \left( \frac{5}{4}B + I - \lambda I \right) = \det \left( \frac{5}{4} I \right) \det \left( B - (\lambda I) \frac{4}{5} I \right) \quad \Rightarrow \quad \det(A + B) = \det(B + I) \\
\text{d) } & \frac{4}{5}(\lambda + 1) = 1, 2, 3, 4 \pm 5i \quad \Rightarrow \quad \lambda = 1 + \frac{5i}{4}, 1 + \frac{10}{4}, 1 + \frac{15}{4}, 1 + \frac{20}{4} + \frac{25i}{4}, 1 + \frac{20}{4} - \frac{25i}{4} \\
\text{e) } & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is an example. Eigenanalysis is routine, not recorded here.}
\end{align*}\]

Staple this page to the top of all Ch6 work. Submit one package per chapter.
Differential Equations and Linear Algebra 2250-2
Final Exam 10:30am 3 May 2005

Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system \( \mathbf{x}' = A \mathbf{x} \), given
\[
A = \begin{bmatrix}
-4 & 1 & 1 \\
1 & -4 & 1 \\
0 & 0 & -4
\end{bmatrix}
\]

☐ [25%] Ch7(b): Solve for \( x(t) \) in the system below. Don’t solve for \( y(t) \)!
\[
\begin{align*}
\mathbf{x}' &= \mathbf{x} + \mathbf{y}, \\
\mathbf{y}' &= -16\mathbf{x} + \mathbf{y}.
\end{align*}
\]

☐ [25%] Ch7(c): State a result for \( n \times n \) systems which displays the solution \( \mathbf{x}(t) \) for \( \mathbf{x}' = A \mathbf{x} \) in terms of the eigenpairs of matrix \( A \). Hypotheses and the solution formula are required.

☐ [25%] Ch7(d): Let \( \mathbf{x}(t) \) be the general solution of the \( 3 \times 3 \) system \( \mathbf{x}' = A \mathbf{x} \). Assume \( A \) has eigenvalues \( \lambda_1 = -1, \lambda_2 = -10, \lambda_3 = -101 \). Prove that each of the three components of \( \mathbf{x}(t) \) have limit zero at \( t = \infty \).

\( a \) \( \mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-5t} \)

\( b \) \( (\lambda - 1)^2 + 16 = 0 \) \( \mathbf{x}(t) = (c_1 \cos 4t + c_2 \sin 4t) e^t \)

\( c \) Let \( A \) have \( n \) eigenpairs \( (\lambda_i, \mathbf{v}_i) \), \( i = 1, \ldots, n \) with \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) independent, then \( \mathbf{x}' = A \mathbf{x} \) has sol
\[
\mathbf{x}(t) = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{\lambda_i t}
\]

\( d \) Each of components \( c_1 \mathbf{v}_1 e^{-t}, c_2 \mathbf{v}_2 e^{101t}, c_3 \mathbf{v}_3 e^{-10t} \) has limit zero at \( t = \infty \), so also does the sum of these, \( \mathbf{x}(t) \).

Staple this page to the top of all Ch7 work. Submit one package per chapter.
Differential Equations and Linear Algebra 2250-2
Final Exam 10:30am 3 May 2005

Ch10. (Laplace Transform Methods)
It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

[30%] Ch10(a): Find \( f(t) \) by partial fraction methods, given
\[
\mathcal{L}(f(t)) = \frac{4s^2 + 12}{(s + 1)(s - 3)(s - 1)^2}.
\]

[30%] Ch10(b): Apply Laplace’s method to find a formula for \( \mathcal{L}(x(t)) \). Do not solve for \( x(t) \)!
Document steps by reference to tables and rules.

\[ x^{iv} + x'' = t^2 + e^{-2t} + e^t \sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1. \]

[35%] Ch10(c): Apply Laplace’s method to the system to find a formula for \( \mathcal{L}(y(t)) \). Find a \( 2 \times 2 \) system for \( \mathcal{L}(x), \mathcal{L}(y) \) [20%]. Solve it only for \( \mathcal{L}(y) \) [15%]. Do not solve for \( x(t) \) or \( y(t) \)!

\[
x'' = 2x + 3y + t^2, \quad y'' = 4x + 3y, \\
x(0) = 1, \quad x'(0) = 4, \quad y(0) = 2, \quad y'(0) = 3.
\]

[35%] Ch10(d): Solve for \( x(t) \), given
\[
\mathcal{L}(x(t)) = \frac{d}{ds} \left( \frac{s - 1}{s^2 - 2s + 17} \right) + \frac{s + 1}{s^2} + \frac{2 + s}{s^2 + 5s} + \mathcal{L}(t + \sin t).
\]

[35%] Ch10(e): Find \( f(t) \), given \( \mathcal{L}(f(t)) = \arctan(1/(s + 2)) \).

\( f(t) = a e^{-t} + b e^{3t} + c e^{-t} + d t e^{t}, \quad a = -1, \quad b = 3, \quad c = -2, \quad d = -4 \)

\( x(t) = p/q, \quad p = \frac{2}{s^2} + \frac{1}{s + 2} + \frac{2}{s^2 + 4}; q = s^2 + 5s \)

\( y(t) = \frac{d}{dt} [A_2(t)], \quad A_2 = \begin{pmatrix} s^2 + 2 & \frac{2}{s + 2} & s + 4 \\ -4 & s^2 + 2s + 3 & s + 2 \end{pmatrix}, \quad A = \begin{pmatrix} s^2 + 2 & -3 \\ -4 & s^2 + 3 \end{pmatrix}, \quad \text{sys} = A \left( \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right) = \cos (A_2(t)) \)

\( x(t) = -t e^t \cos (4t) + 1 + t + \frac{2}{s} + \frac{3}{5} e^{-5t} + t + \sin (t) \)

Staple this page to the top of all Ch10 work. Submit one package per chapter.