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Differential Equations and Linear Algebra 2250-1

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Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **fourth entry on the third row** of the inverse matrix B^{-1} by the formula $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of 3×3 Sarrus' rule is disallowed (2×2 use is allowed).

$$B = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

☐ [25%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has a unique solution. Do not solve for $\mathbf{x}$!

$$R = \begin{bmatrix} 2 & 1 & 2k \\ 2 & -4k & -2 \\ 0 & 0 & 4k \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1 + 4\sin(\pi k) \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Let A be a 41×41 triangular matrix with diagonal entries $a_{jj} = \sin(j - 21)$, $j = 1, \ldots, 41$. Prove that $A\mathbf{x} = \mathbf{0}$ has a infinitely many solutions $\mathbf{x}$.

☐ [25%] Ch3(d): Let A be a 41×42 triangular matrix ($a_{ij} = 0$ for $i > j$) with coefficients either 0 or 1. Find infinitely many solutions $\mathbf{x}$ for $A\mathbf{x} = \mathbf{0}$.

☐ [25%] Ch3(e): An example exists of a square matrix A such that A^3 is not the zero matrix but $A^4 = 0$. Prove that the example is not a diagonal matrix.

ⓐ $C = B^{-1}$, $c_{34} = \dfrac{\mathrm{cof}(B,4,3)}{\det(B)}$, $\det(B) = 3\begin{vmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & -1 & 2 \end{vmatrix} = 3(2) = 6$,

$\mathrm{cof}(B,4,3) = (-1)^{4+3}\,\mathrm{minor}(B,4,3) = -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 0$. Then $\boxed{c_{34} = 0}$.

ⓑ Unique $\iff \det(R) \ne 0 \iff 4k(-8k-2) \ne 0 \iff \boxed{k \ne 0 \text{ and } k \ne -1/4}$

ⓒ square systems $A\vec{x} = \vec{0}$ have ∞-many solutions $\iff \det(A) = 0$ $\iff$ product of diag elements of $A = 0$. Since $a_{jj} = 0$ for $j = 21$, Then $\det(A) = 0$, Then $A\vec{x} = \vec{0}$ has ∞-many solutions.

ⓔ Suppose A is diagonal, $A = \mathrm{diag}(a_1, \ldots, a_n)$. Then $A^k = \mathrm{diag}(a_1^k, \ldots, a_n^k)$. If $A^4 = 0$, Then $a_1 = a_2 = \cdots = a_n = 0$. Hence $A^3 \ne 0$ can't happen. So, A diagonal cannot satisfy $A^2 \ne 0$ and $A^3 = 0$.

ⓓ There is at least one free variable, because $\mathrm{rank}(A) \le 41$. Choose $t_1 = 1, 2, 3, \ldots$ in the gen sol obtained from $\mathrm{rref}(A)$. Then ∞-many sols are found.

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}3$ in $\mathcal{R}^4$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \\ 4 \end{pmatrix}.$$

☐ [60%] Ch4(b): Let A be a 20×20 matrix. Define V to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^{20}$ such that $A(A-I)\mathbf{x} + (2I - A)(I + A)\mathbf{x} = \mathbf{x}$. Prove that V is a subspace of $\mathcal{R}^{20}$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of the 4×4 matrix A below [30%] and (2) the row space of the 4×4 matrix A below [30%]. The two displayed bases must consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -3 & 1 \\ 0 & 0 & 3 & 1 \\ 2 & -2 & 4 & 1 \end{pmatrix}$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{4x^2 - 12x + 4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

(a) Test: $\text{rank}(\text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3)) = 3 \iff \vec{v}_1, \vec{v}_2, \vec{v}_3$ independent

Let $A = \begin{pmatrix} -1 & 3 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 6 \\ 1 & 1 & 4 \end{pmatrix}$. Then $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\text{rank}(A) = 2 \Rightarrow$ $\boxed{\text{dependent}}$

(b) Let $B = A(A-I) + (2I - A)(I + A) - I$. Then $V = \{\vec{x} : B\vec{x} = \vec{0}\}$.

By superposition of matrices, $\vec{v}_1, \vec{v}_2$ in V implies $c_1\vec{v}_1 + c_2\vec{v}_2$ in V.

Also, $B\vec{0} = \vec{0}$ implies $\vec{0}$ in V. The Subspace Criterion applies; V is a subspace

(c) $\text{rref}(A) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\text{rref}(A^T) = \begin{pmatrix} 7 & 0 & 0 & 13 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{7} \Rightarrow$ $\boxed{\begin{array}{l} \text{Cols 1, 3, 4 indep; basis} \\ \text{rows 1, 2, 3 indep; basis} \end{array}}$

(d) $\begin{cases} -4 = b(4) \\ 20 = d(4) \\ 0 = a + c \\ 4 = -a + b + c + d \end{cases}$ Then $\boxed{a = 0,\ b = -1,\ c = 0,\ d = 5}$

Second proof of (b): $V = \{\vec{x} : 2I\vec{x} = \vec{x}\} = \{\vec{0}\}$ = subspace by Subspace Criterion.

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Ch5. (Linear Equations of Higher Order)

☐ [30%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

1.[15%] $(r^2-5r)^2(r^2+25)=0$,
2.[15%] $(r+4)^2(r^2+2r+2)^3(r^2-16)^2=0$

☐ [30%] Ch5(b): Given a damped spring-mass system $mx''(t)+cx'(t)+kx(t)=0$ with $m=6$, $c=13$ and $k=6$, solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

☐ [40%] Ch5(c): Determine the **final form** of a trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv}+16y''+8y=x^2(1+e^{2x})+x\sin 2x+2\sin x\cos x$$

☐ [30%] Ch5(d): Find the steady-state periodic solution for the equation

$$x''+2x'+65x=7\cos(5t).$$

(a) roots $=0,0,5,5,5i,-5i$ $\quad y=c_1+c_2x+(c_3+c_4x)e^{5x}+c_5\cos 5x+c_6\sin 5x$

roots $=-4,-4,-4,-4,4,4,-1\pm i,-1\pm i,-1\pm i$ $\quad y=(c_1+c_2x+c_3x^2+c_4x^3)e^{-4x}+(c_5+c_6x)e^{4x}+(c_7+c_8x+c_9x^2)e^{-x}\cos x+(c_{10}+c_{11}x+c_{12}x^2)e^{-x}\sin x$

(b) $(3r+2)(2r+3)=6r^2+13r+6=0$ $\quad y=c_1e^{-2x/3}+c_2e^{-3x/2}$ $\quad$ over-damped

(c) $r^4+16r^2+8=(r^2+8)^2-56=(r^2+8-\sqrt{56})(r^2+8+\sqrt{56})$ Two complex conjugate roots with atoms $\cos\omega_1x$, $\sin\omega_1x$, $\cos\omega_2x$, $\sin\omega_2x$ where $\omega_1^2=8-\sqrt{56}$, $\omega_2^2=8+\sqrt{56}$. These atoms are unrelated to atoms of the RHS, so the fixup rule does not apply. Then

$y=$ linear combination of atoms $1,x,x^2,e^{2x},xe^{2x},x^2e^{2x},\cos 2x,\sin 2x,x\cos 2x,x\sin 2x$

The term $2\sin x\cos x=\sin 2x$ is already included. There are 10 constants in y.

(d) $x(t)=A\cos(5t)+B\sin(5t)$, $A=\dfrac{14}{85}$, $B=\dfrac{7}{170}$

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Ch6. (Eigenvalues and Eigenvectors)

☐ [30%] Ch6(a): Find the eigenvalues of the matrix A:

$$A = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

☐ [35%] Ch6(b): Let A be a 2×2 matrix with eigenpairs

$$(\pi, \mathbf{v}_1), \quad (2\pi, \mathbf{v}_2).$$

Display formulae for two distinct diagonal matrices D_1, D_2 and two invertible 2×2 matrices P_1, P_2 such that $AP_1 = P_1D_1$ and $AP_2 = P_2D_2$ [15%]. Justify your claims [20%].

☐ [35%] Ch6(c): Assume $\det(A - \lambda I) = \det(B - \lambda I)$ for two 3×3 matrices A, B. Let A have eigenvalues 3, 4, 5. Find the eigenvalues of $(1/4)B + 3I$, where I is the identity matrix.

☐ [35%] Ch6(d): An example exists of a 3×3 matrix C which has eigenvalues λ_1, λ_2, λ_3 but only two independent eigenvectors. Display an example C where $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 1$. Justify by doing the eigenanalysis of C.

(a) $\lambda = 3, 3, 1, 6 \qquad \det(A - \lambda I) = (3-\lambda)(3-\lambda)(\lambda^2 - 7\lambda + 6)$

(b) $D_1 = \mathrm{diag}(\pi, 2\pi)$, $D_2 = \mathrm{diag}(2\pi, \pi)$, $P_1 = \mathrm{aug}(v_1, v_2)$, $P_2 = \mathrm{aug}(v_2, v_1)$. From Theory $AP = PD$ we obtain $AP_1 = P_1D_1$, $AP_2 = P_2D_2$.

(c) $\det(\frac{1}{4}B + 3I - \lambda I) = \det((\frac{1}{4}I)(B + 12I - 4\lambda I))$
$= \det(\frac{1}{4}I)\det(B - (4\lambda - 12)I)$
$= \det(\frac{1}{4}I)\det(A - (4\lambda - 12)I)$

Then $4\lambda - 12 = 3, 4, 5 \Rightarrow \boxed{\lambda = \frac{15}{4}, \frac{16}{4}, \frac{17}{4}}$

(d) Let $C = \begin{pmatrix} 2 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{pmatrix}$. We want only one eigenvector for $\lambda = 2$. Then $C - 2I = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}$. To get exactly one free variable we need $C - 2I \approx \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to have $a \neq 0$. So take $a = 1$, $b = c = 0$. Then $C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then eigenpairs are $\left(1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$ and $\left(2, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$.

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Ch7. (Linear Systems of Differential Equations)

☐ [50%] Ch7(a): Apply the eigenanalysis method to solve the system $\mathbf{x}' = A\mathbf{x}$, given

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

☐ [25%] Ch7(b): Solve for $x(t)$ in the system below. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + 3y, \\ y' &= -3x + y. \end{aligned}$$

☐ [25%] Ch7(c): State a result for 4×4 systems which displays the solution $\mathbf{x}(t)$ for $\mathbf{x}' = A\mathbf{x}$ in terms of the eigenpairs of matrix A. Hypotheses and the solution formula are required.

☐ [25%] Ch7(d): Let $\mathbf{x}(t)$ be the general solution of the 3×3 system $\mathbf{x}' = A\mathbf{x}$. Assume A has eigenvalues $\lambda_1 = -0.1$, $\lambda_2 = -0.001$, $\lambda_3 = -0.02$. Prove that each of the three components of $\mathbf{x}(t)$ have limit zero at $t = \infty$.

(a) $\lambda = -2, -4, -4$ $\quad \vec{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}-1\\1\\0\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad \vec{x}(t) = c_1\vec{v}_1 e^{-2t} + c_2\vec{v}_2 e^{-4t} + c_3\vec{v}_3 e^{-4t}$

(b) $\det\left(\begin{pmatrix}1 & 3\\-3 & 1\end{pmatrix} - \lambda I\right) = 0$ has roots $1 \pm 3i \Rightarrow x(t) = (c_1 \cos 3t + c_2 \sin 3t)e^{t}$

(c) Let A be real 4×4 with eigenpairs $(\lambda_i, \vec{v}_i)$, $i = 1,2,3,4$ and $\{\vec{v}_1, \ldots, \vec{v}_4\}$ independent. Then $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + c_3\vec{v}_3 e^{\lambda_3 t} + c_4\vec{v}_4 e^{\lambda_4 t}$

(d) The general solution $\vec{x}(t)$ is the sum of terms $c_1\vec{v}_1 e^{-t/10} + c_2\vec{v}_2 e^{-t/1000} + c_3\vec{v}_3 e^{-t/50}$. Each term has limit zero at $t=\infty$, due to the negative exponential factor. Thus, $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$.

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Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic Laplace integral table and know the basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

☐ [30%] Ch10(a): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2+24}{(s-1)(s-3)(s+1)^2}.$$

☐ [30%] Ch10(b): Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. Do not solve for $x(t)$! Document steps by reference to tables and rules.

$$x^{iv} + 4x'' = t^3 + te^{-2t} + e^t \sin 2t, \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = 2.$$

☐ [35%] Ch10(c): Apply Laplace's method to the system to find a formula for $\mathcal{L}(y(t))$. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [20%]. Solve it **only** for $\mathcal{L}(y)$ [15%]. Do not solve for $x(t)$ or $y(t)$!

$$\begin{aligned} x'' &= 2x + 5y, \\ y'' &= 2x + 4y, \\ x(0) &= 1, \quad x'(0) = 3, \\ y(0) &= 2, \quad y'(0) = 4. \end{aligned}$$

☐ [35%] Ch10(d): Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\frac{2}{s^2-2s+5}\right) + \frac{s^2}{(s+1)^3} + \frac{1+s}{s^2+5s} - \frac{d}{ds}\mathcal{L}(\sin t).$$

☐ [35%] Ch10(e): Find $f(t)$, given $\mathcal{L}(f(t)) = \arctan(1/(s+3))$.

(a) $\mathcal{L}(f(t)) = \mathcal{L}(ae^t + be^{3t} + ce^{-t} + dte^{-t})$ and $a=-4$, $b=3$, $c=1$, $d=4$

(b) $\mathcal{L}(x) = p/q$, $p = \mathcal{L}(t^3 + te^{-2t} + e^t \sin 2t) + 2 = \frac{6}{s^4} + \frac{1}{(s+2)^2} + \frac{2}{(s-1)^2+4} + 2$
and $q = s^4 + 4s^2$

(c) $\begin{cases} 2X + (4-s^2)Y = -4-2s \\ (2-s^2)X + 5Y = -3-s \end{cases}$ $\mathcal{L}(x) = X = \frac{P_1}{q_1}$, $P_1 = 8+6s+s^3+3s^2$, $q_1 = -2+s^4-6s^2$
$\mathcal{L}(y) = Y = \frac{P_2}{q_1}$, $P_2 = 2(-3-s) - (4-s^2)(2-s^2)$

(d) $\mathcal{L}(x(t)) = \frac{d}{ds}\left(\frac{2}{(s-1)^2+4}\right) + \left.\frac{(s-1)^2}{s^3}\right|_{s\to(s+1)} + \frac{1/5}{s} + \frac{4/5}{s+5} + \mathcal{L}(t\sin t)$
$= \mathcal{L}(-te^t \sin 2t) + \left.\left(\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s^3}\right)\right|_{s+1} + \frac{1/5}{s} + \frac{4/5}{s+5} + \mathcal{L}(t\sin t)$
$= \mathcal{L}\left(-te^t \sin 2t + (1-2t+\frac{t^2}{2})e^{-t} + 1/5 + 4/5\, e^{-5t} + t\sin t\right)$ Then apply Lerch's thm.

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(e) Let $u = \mathcal{L}(f(t))$. Then $\frac{du}{ds} = \frac{-1/(s+3)^2}{1 + 1/(s+3)^2} = \frac{-1}{(s+3)^2+1} = \mathcal{L}(-e^{-3t}\sin t)$. Then $f = \frac{-e^{-3t}\sin t}{-t}$