Calculus III 2210-4
Final Exam
Monday 12 December 2005

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Mostly blank solutions will count 40% of their assigned value.

Name. KEY

2210-4 FINAL EXAM

1. (Derivatives) Complete the following.

(a) [50%] Find f_{xy} for $f(x,y) = (x^3 + y^2)(x - y)^2$.

(b) [50%] Find the gradient of $f(x,y,z) = (2x + 3y + 4z)^2 e^{3x+4y+5z}$ at $x = y = 0$, $z = 1$.

(a) $f_x = 3x^2(x-y)^2 + 2(x^3+y^2)(x-y)$

$f_{xy} = 3x^2(2(x-y))(-1) + 4y(x-y) + 2(x^3+y^2)(-1)$

$-8x^3 + 6x^2y + 4xy - 6y^2$

(b) $\mathrm{grad}(f) = \begin{pmatrix} 2(2x+3y+4z)(2)e^{3x+4y+5z} + 3f \\ 2(2x+3y+4z)(3)e^{3x+4y+5z} + 4f \\ 2(2x+3y+4z)(4)e^{3x+4y+5z} + 5f \end{pmatrix}$

$= \begin{pmatrix} 16e^5 + 48e^5 \\ 24e^5 + 64e^5 \\ 32e^5 + 80e^5 \end{pmatrix} = \begin{pmatrix} 64 \\ 88 \\ 112 \end{pmatrix} e^5$

4 min

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2. (Chain rule) Complete the following. Leave answers with symbols and don't expand.

(a) [50%] Let $w = xy\sin(x+y)$, $x = 2t$, $y = 3t^2$, $z = 4t$. Find dw/dt.

(b) [50%] Let $w = u^2 v$, $u = x^2 + 2xy$, $v = xyz$. Find the partials of w in variables x, y, z.

(a)
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

ok → $= [y\sin(x+y) + xy\cos(x+y)](2) + [x\sin(x+y) + xy\cos(x+y)](6t) + 0$

also ok → $= 2[3t^2\sin(2t+3t^2) + 6t^3\cos(2t+3t^2)] + 6t[2t\sin(2t+3t^2) + 6t^3\cos(2t+3t^2)]$

also
$$\boxed{\frac{dw}{dt} = 18t^2\sin(u) + 12t^3\cos(u) + 36t^4\cos(u),\ u = 2t+3t^2}$$

(b)
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$
$$= 2uv(2x+2y) + u^2 yz$$
$$\frac{\partial w}{\partial y} = \begin{pmatrix} 2uv \\ u^2 \end{pmatrix} \cdot \begin{pmatrix} 2x \\ xz \end{pmatrix}$$
$$= 4uvx + u^2 xz$$
$$\frac{\partial w}{\partial z} = \begin{pmatrix} 2uv \\ u^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ xy \end{pmatrix}$$
$$= u^2 xy$$

4 min

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3. **(Gradient)** Complete the following.

(a) [30%] Find the directional derivative of $f(x, y, z) = x^2y^3 - xyz^2$ at $(-2, 1, 0)$ in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(b) [70%] Find a point on the surface $x^2+2y^2+3z^2 = 12$ where the tangent plane is perpendicular to the line through $(1, 3, 2)$ with direction $2\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}$.

(a) $\text{grad}(f) = \begin{pmatrix} 2xy^3 - yz^2 \\ 3x^2y^2 - xz^2 \\ -2xyz \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \\ 0 \end{pmatrix}$

$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}$ a unit vector

D.D $= \text{grad}(f) \cdot \vec{u}$

$= \boxed{\frac{20}{\sqrt{6}}}$

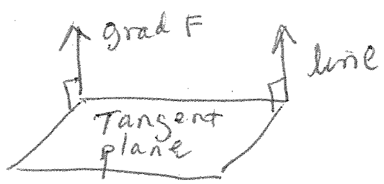
(b) Surface is $F = x^2+2y^2+12z^2-12 = 0$; $\text{grad}(F) = \begin{pmatrix} 2x \\ 4y \\ 6z \end{pmatrix}$

Line tangent is $\begin{pmatrix} 2 \\ 8 \\ -6 \end{pmatrix}$.

Want grad(F) parallel to the line tangent.

An easy choice is $\text{grad}(F) = \begin{pmatrix} 2 \\ 8 \\ -6 \end{pmatrix}$ which gives

$\boxed{\text{point} = (1, 2, -1)}$



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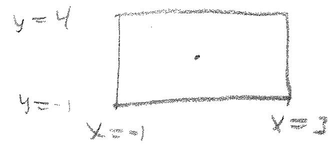
4. (Maxima and Minima) Complete **one** of the following.

(a) [50%] Find the global maximum and global minimum of $f(x,y) = x^2 + y^2$ on the rectangle $-1 \le x \le 3$, $-1 \le y \le 4$.

(b) [50%] Find the maximum of $f(x,y) = 4x^2 - 4xy + y^2$ on the circle $x^2 + y^2 = 1$.

(a) $\boxed{\text{Global min} = 0 \text{ at } (0,0)}$



7 min

along edges,

$x=-1$, $f = 1+y^2$ max is 17

$x=3$, $f = 9+y^2$ max is 25

$y=-1$, $f = x^2+1$ max is 10

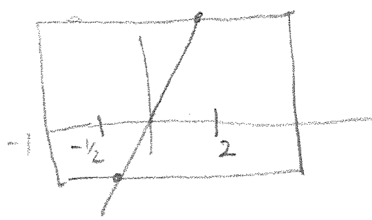
$y=4$, $f = x^2+16$ max is 25

on interior, $\operatorname{grad}(f) = \begin{pmatrix}0\\0\end{pmatrix}$ gives local extrema

$$\begin{pmatrix}8x-4y\\-4x+2y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$y = 2x$ is only condition, but

$-1 \le x \le 3$ and $-1 \le y \le 4$



$f = x^2 + 4x^2$

$= 5x^2$

max on $-\frac{1}{2} \le x \le 2$

is 20

$\boxed{\text{Global max} = 25 \text{ on boundary}}$

(b) $f = 4x^2 - 4xy + y^2$

10 min

$g = x^2 + y^2 - 1$

$\nabla f = \begin{pmatrix}8x-4y\\-4x+2y\end{pmatrix}$

$\nabla g = \begin{pmatrix}2x\\2y\end{pmatrix}$

$\nabla f = \lambda \nabla g$

$$\begin{cases}8x-4y = 2\lambda x\\-4x+2y = 2\lambda y\end{cases} \iff \begin{cases}4(2x-y) = 2\lambda x\\2(-2x+y) = 2\lambda y\end{cases}$$

$2\lambda x = -4\lambda y$

$\lambda(2x+4y) = 0$

Either $\lambda = 0$ or else $x = -2y$

If $\lambda = 0$, then $y = 2x$.

Case 1 $x = -2y$

$x^2 + y^2 = 1$

$5y^2 = 1$

$y = \pm 1/\sqrt{5}$

$f = 3x^2 + 8y^2 + 1$

$= 20y^2 + 1$

$= 5$

Case 2 $y = 2x$

$x^2 + y^2 = 1$

$5x^2 = 1$

$x = \pm 1/\sqrt{5}$

$\longrightarrow$ Since $f = 3x^2 - 4xy + 1$ on $g = 0$

Then $f = 3x^2 - 8x^2 + 1$

$= 1 - 5x^2$

$= 0$

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$\boxed{\text{max} = 5 \text{ at } (2/\sqrt{5}, -1/\sqrt{5})}$

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5. (Double Integrals) Complete the following.

(a) [50%] Let $f(x,y) = 1$ on R_1, $f(x,y) = 3$ on R_2, where R_1 and R_2 are two regions that don't intersect. Suppose each region has area 2. Let R be the region consisting of R_1 and R_2. Find $\int\int_R (2f(x,y) + \ln(f(x,y)))dA$.

(b) [50%] Let R be the rectangle defined by $0 \le x \le 2$, $1 \le y \le 3$. Divide R into 4 equal sub-rectangles R_1, R_2, R_3, R_4. Write out explicitly the Riemann sum for this subdivision of R, corresponding to the integral $\int\int_R g(x,y)dA$. Use symbols to save time. Draw a figure showing the sub-rectangles. Explain all symbols used.

(a) $\iint_R F\,dA = \iint_{R_1} F\,dA + \iint_{R_2} F\,dA$

2 min

$= F_1 \,\text{area}(R_1) + F_2 \,\text{area}(R_2)$ where F_1, F_2 are constant

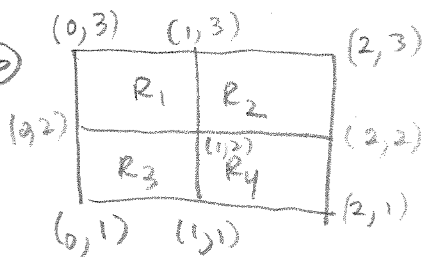
$= 2(F_1 + F_2)$ $F_1 = 2 + \ln 1$

$= \boxed{2(8 + \ln 3)}$ $= 2$

$F_2 = 6 + \ln 3$

(b)

3 min



R.S. $= g(C_1)\,\text{area}(R_1) + g(C_2)\,\text{area}(R_2) + \cdots$

$= g(C_1) + g(C_2) + g(C_3) + g(C_4)$

where C_1, C_2, C_3, C_4 are the centers of the rectangles

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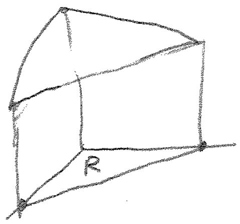
6. (Double Integrals) Complete **one** of the following.

(a) [100%] Find the volume of the solid in the first octant bounded by the surface $z = e^{x-y}$, the plane $x + y = 1$ and the coordinate planes.

3 min (b) [100%] Let R be the planar xy-region described by $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$. Let $f(x,y) = 1/\sqrt{4 - x^2 - y^2}$. Evaluate $\int\int_R f(x,y)dA$ using polar coordinates.

(a)



$$\text{vol} = \iint_R e^{x-y}\,dA$$

$$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$$

$$\text{vol} = \int_0^1 \int_0^{1-x} e^{x-y}\,dy\,dx$$

$$= \int_0^1 -e^{x-y}\Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 (-e^{x-1+x} + e^x)\,dx$$

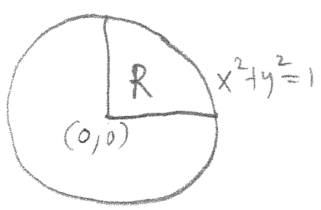
$$= -\frac{e^{2x-1}}{2} + e^x \Big|_{x=0}^{x=1}$$

$$= -\frac{e}{2} + e + \frac{e^{-1}}{2} - 1$$

$$= \boxed{\frac{e}{2} + \frac{e^{-1}}{2} - 1}$$

(b)

6 min



$$R^* = \{(r,\theta) : 0 \le \theta \le \pi/2,\ 0 \le r \le 1\}$$

$$\iint_R f\,dA = \iint_{R^*} f\,r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{\pi/2} \frac{r}{(4-r^2)^{1/2}}\,d\theta\,dr$$

$$= \frac{\pi}{2} \int_0^1 \frac{r\,dr}{(4-r^2)^{1/2}}$$

$$= \frac{\pi}{2} \int \frac{-du/2}{u^{1/2}} \qquad u = 4 - r^2 \quad du = -2r\,dr$$

$$= -\frac{\pi}{2} \frac{u^{1/2}}{1/2} \Big|_0^1$$

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$$= \boxed{\pi(\sqrt{4} - \sqrt{3})}$$

Name. KEY

7. (Surface Area)
Find the area of the part of the conical surface $x^2 + y^2 = z^2$ that is directly above the xy-plane triangle with vertices $(0,0)$, $(4,0)$ and $(0,4)$. **Display all integration steps and include a figure.**

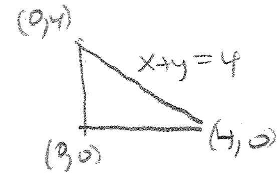
4 min

$$\text{area} = \iint_S dS$$

$$= \iint_R \sqrt{f_x^2+f_y^2+1}\, dA$$

$$z = f(x,y)$$
$$= \sqrt{x^2+y^2}$$

$R =$ triangle



$$R = \{(x,y):\ 0 \le x \le 4,\ 0 \le y \le 4-x\}$$

$$f_x = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)$$
$$= \frac{x}{f}$$
$$f_y = \frac{y}{f}$$
$$\sqrt{f_x^2+f_y^2+1} = \sqrt{\frac{x^2+y^2}{f^2}+1}$$
$$= \sqrt{2}$$

$$\text{area} = \int_0^4 \int_0^{4-x} \sqrt{2}\, dy\, dx$$

$$= \int_0^4 \sqrt{2}(4-y)\, dx$$

$$= -\sqrt{2}\frac{(4-x)^2}{2}\Big|_0^4$$

$$= \boxed{8\sqrt{2}}$$

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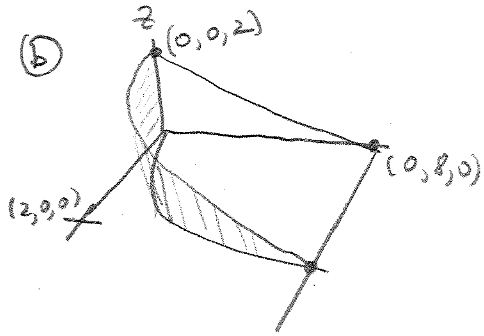
8. (Triple Integrals) Complete **one** of the following.

(a) [100%] Evaluate $\int_{-2}^{4}\int_{x-1}^{x+1}\int_{0}^{\sqrt{2y/x}} 3xyz\,dz\,dy\,dx$.

(b) [100%] Find by using triple integration the volume of the solid in the first octant bounded by $y = 2x^2$ and $y + 4z = 8$. Draw a figure. Display all steps.

(a)
$$\int_{-2}^{4}\int_{x-1}^{x+1} 3xy\left(\frac{z^2}{2}\right)\Bigg|_{z=0}^{z=\sqrt{\frac{2y}{x}}} dy\,dx$$
$$= \int_{-2}^{4}\int_{x-1}^{x+1} \frac{3}{2}xy\left(\frac{2y}{x}\right)dy\,dx$$
$$= \int_{-2}^{4}\int_{x-1}^{x+1} 3y^2\,dy\,dx$$
$$= \int_{-2}^{4}\left(y^3\Big|_{y=x-1}^{y=x+1}\right)dx$$
$$= \int_{-2}^{4}\left((x+1)^3-(x-1)^3\right)dx$$
$$= \left(\frac{(x+1)^4}{4}-\frac{(x-1)^4}{4}\right)\Bigg|_{-2}^{4}$$
$$= \frac{5^4}{4}-\frac{3^4}{4}-\frac{1}{4}+\frac{3^4}{4}$$
$$= \boxed{\frac{5^4-1}{4}}$$
$$= 156$$

(b)



$$V = \{(x,y,z) : 0 \le x \le 2,\ 2x^2 \le y \le 8,\ 0 \le z \le (8-y)/4\}$$

$$\text{Vol} = \iiint_V dz\,dy\,dx$$
$$= \int_0^2\int_{2x^2}^{8}\int_0^{2-y/4} dz\,dy\,dx$$
$$= \int_0^2\int_{2x^2}^{8}\left(2-\frac{y}{4}\right)dy\,dx$$
$$= \int_0^2 \left(2y-\frac{y^2}{8}\right)\Bigg|_{2x^2}^{8} dx$$
$$= \int_0^2\left(16-\frac{64}{8}-4x^2+\frac{4x^4}{8}\right)dx$$
$$= \int_0^2\left(8-4x^2+\frac{x^4}{2}\right)dx$$
$$= \left(8x-\frac{4x^3}{3}+\frac{x^5}{10}\right)\Bigg|_0^2$$
$$= \boxed{16-\frac{32}{3}+\frac{32}{10}}$$

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9. (Line Integrals)
(a) [25%] Define work using line integrals. Explain how a line integral equals an ordinary calculus I integral.
(b) [75%] Find the work done by vector force $\mathbf{F} = (2x - y)\mathbf{i} + (2z)\mathbf{j} + (y - z)\mathbf{k}$ where path C is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

2 min

(a) work done by variable force $\vec{F}$ along path C is

$$\text{work} = \int_C \vec{F} \cdot d\vec{r}$$

Here, $\vec{r}(t)$ is a parameterization of curve C, $a \le t \le b$.
In terms of ordinary integrals of Calculus I,

$$\text{work} = \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\right) dt$$

2 min

(b) $\vec{r}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$ $0 \le t \le 1$

$$\text{work} = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_0^1 \begin{pmatrix} 2x-y \\ 2z \\ y-z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt$$
$$= \int_0^1 \begin{pmatrix} t \\ 2t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dt$$
$$= \int_0^1 3t\,dt$$
$$= \frac{3}{2}t^2 \Big|_0^1$$
$$= \boxed{3/2}$$

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10. (Divergence and Curl) Complete the following.

(a) [25%] Define divergence and curl.

(b) [25%] Compute the divergence of the vector function $\mathbf{F} = xyz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.

(c) [40%] Compute the curl of the vector function $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$.

(d) [10%] What does divergence measure, in the case of a vector field $\mathbf{F}$ which represents the velocity field of a fluid?

3 min

(a) $\mathrm{div}(\vec{F}) = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$ where $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$

$\mathrm{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$

(b) $\mathrm{div}(\vec{F}) = yz + 0 + 0$

$= \boxed{yz}$

(c) $\mathrm{curl}(\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y+z & x+z & x+y \end{vmatrix}$

$= \begin{pmatrix} 1-1 \\ -(1-1) \\ 1-1 \end{pmatrix}$

$= \boxed{\vec{0}}$

(d) It measures the rate of creation and destruction of fluid.

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