Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Mostly blank solutions will count 40% of their assigned value.
1. (Derivatives) Complete the following.
   (a) [50%] Find $f_{xy}$ for $f(x, y) = (x^3 + y^2)(x - y)^2$.
   (b) [50%] Find the gradient of $f(x, y, z) = (2x + 3y + 4z)^2e^{3x+4y+5z}$ at $x = y = 0, z = 1$.

   (a) $f_x = 3x^2(x-y)^2 + 2(x^2+y^2)(x-y)$

   $f_{xy} = 3x^2(2(x-y)(-1) + 4y(x-y) + 2(x^2+y^2)(-1)$

   (b) $\nabla f = \begin{pmatrix} 2(2x+3y+4z)(2)e^{3x+4y+5z}\frac{\partial}{\partial x} + 3f \\ 2(2x+3y+4z)(3)e^{3x+4y+5z}\frac{\partial}{\partial y} + 4f \\ 2(2x+3y+4z)(4)e^{3x+4y+5z}\frac{\partial}{\partial z} + 5f \end{pmatrix}$

   $= \begin{pmatrix} 16e^5 + 48e^5 \\ 24e^5 + 64e^5 \\ 32e^5 + 80e^5 \end{pmatrix} = \begin{pmatrix} 64 \\ 88 \\ 112 \end{pmatrix}e^5$
2. (Chain rule) Complete the following. Leave answers with symbols and don’t expand.
   (a) [50%] Let \( w = xy \sin(x + y) \), \( x = 2t \), \( y = 3t^2 \), \( z = 4t \). Find \( \frac{dw}{dt} \).
   (b) [50%] Let \( w = u^2v \), \( u = x^2 + 2xy \), \( v = xyz \). Find the partials of \( w \) in variables \( x \), \( y \), \( z \).

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

\[
= \left[ y \sin(x+y) + xy \cos(x+y) \right] (2t) + \left[ x \sin(x+y) + xy \cos(x+y) \right] (6t) + 0
\]

Also, \( \frac{dw}{dt} = 18t^2 \sin(x) + 12t^2 \cos(y) + 36ty \cos(x) \sin(y) \) \( u = 2t + 2t^2 \)

\[
\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x}
\]

\[
= 2uv (2x + 2y) + u^2 y z
\]

\[
\frac{\partial w}{\partial y} = \begin{pmatrix} 2uv \\ u^2 \end{pmatrix} \begin{pmatrix} 2x \\ xy \end{pmatrix}
\]

\[
= 4uvx + u^2 x z
\]

\[
\frac{\partial w}{\partial z} = \begin{pmatrix} 2uv \\ u^2 \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix}
\]

\[
= u^2 xy
\]

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3. (Gradient) Complete the following.
   (a) [30%] Find the directional derivative of \( f(x, y, z) = x^2y^3 - xyz^2 \) at \((-2, 1, 0)\) in the direction of \(i + 2j + k\).
   (b) [70%] Find a point on the surface \( x^2 + 2y^2 + 3z^2 = 12 \) where the tangent plane is perpendicular to the line through \((1, 3, 2)\) with direction \(2i + 8j - 6k\).

\[
\text{grad}(f) = \begin{pmatrix} 2xy^3 - yz^2 \\ 3x^2y^2 - xz^2 \\ -2xyz \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \\ 0 \end{pmatrix}
\]

\[
\mathbf{u} = \left( \frac{1}{2}, \frac{1}{2} \right), \quad \frac{1}{\sqrt{6}} \quad \text{a unit vector}
\]

\[
D \cdot D = \text{grad}(f) \cdot \mathbf{u} = \frac{20}{\sqrt{6}}
\]

(b) Surface is \( F = x^2 + 2y^2 + 12z^2 - 12 = 0 \); \( \text{grad}(F) = \begin{pmatrix} 2x \\ 4y \\ 6z \end{pmatrix} \)

Line tangent is \( \begin{pmatrix} \frac{2}{8} \\ \frac{8}{8} \\ -6 \end{pmatrix} \).

Want \( \text{grad}(F) \) parallel to the line tangent.

An easy choice is \( \text{grad}(F) = \begin{pmatrix} \frac{2}{8} \\ \frac{8}{8} \\ -6 \end{pmatrix} \) which gives

Point \( = (1, 2, -1) \)
4. (Maxima and Minima) Complete one of the following:
   (a) [50%] Find the global maximum and global minimum of \( f(x, y) = x^2 + y^2 \) on the rectangle 
       \(-1 \leq x \leq 3, -1 \leq y \leq 4\).
   (b) [50%] Find the maximum of \( f(x, y) = 4x^2 - 4xy + y^2 \) on the circle \( x^2 + y^2 = 1 \).

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(a) \[ \text{Global min. = 0 at (0, 0)} \]

along edges,
\[
\begin{align*}
x = -1, & \quad f = 1 + y^2, \quad \text{max is 17} \\
x = 3, & \quad f = 9 + y^2, \quad \text{max is 25} \\
y = -1, & \quad f = x^2 + 1, \quad \text{max is 10} \\
y = 4, & \quad f = x^2 + 16, \quad \text{max is 25} \\
\end{align*}
\]

at interior, \( \nabla f = (0) \) gives local extrema
\[
\begin{pmatrix}
8x-4y \\
-4x+2y
\end{pmatrix} = (0)
\]
\[ y = 2x \quad \text{is only condition, but} \]
\[ f = x^2 + 4x^2 = 5x^2 \]
\[ \text{max on } -\frac{1}{2} \leq x \leq 2 \]
\[ \text{is 20} \]

\[ \text{Global max = 25 on boundary} \]

(b) \[ f = 4x^2 - 4xy + y^2 \]
   \[ \nabla f = \lambda \nabla g \]

\[ g = x^2 + y^2 - 1 \]

\[ \nabla f = \begin{pmatrix} 8x - 4y \\ -4x + 2y \end{pmatrix} \]

\[ \nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \]

\[ \begin{align*}
8x - 4y &= 2x \\
-4x + 2y &= 2y \\
\end{align*} \]

\[ 6x = 4 \lambda y \]

\[ \lambda (2x + 4y) = 0 \]

Either \( \lambda = 0 \) or \( x = -2y \)

\[ \begin{align*}
\text{Case 1: } & \quad x = -2y \\
& \quad x^2 + y^2 = 1 \\
& \quad 5y^2 = 1 \\
& \quad \frac{y}{\sqrt{5}} = 1 \\
\end{align*} \]

\[ \nabla f = 3x^2 + 8y^2 + 1 \quad \text{at } \left( \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right) \]

\[ = 5 \]

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5. (Double Integrals) Complete the following.
(a) [50%] Let \( f(x, y) = 1 \) on \( R_1 \), \( f(x, y) = 3 \) on \( R_2 \), where \( R_1 \) and \( R_2 \) are two regions that don't intersect. Suppose each region has area 2. Let \( R \) be the region consisting of \( R_1 \) and \( R_2 \). Find \( \int \int_R (2f(x, y) + \ln(f(x, y))) \, dA \).
(b) [50%] Let \( R \) be the rectangle defined by \( 0 \leq x \leq 2 \), \( 1 \leq y \leq 3 \). Divide \( R \) into 4 equal sub-rectangles \( R_1, R_2, R_3, R_4 \). Write out explicitly the Riemann sum for this subdivision of \( R \), corresponding to the integral \( \int \int_R g(x, y) \, dA \). Use symbols to save time. Draw a figure showing the sub-rectangles. Explain all symbols used.

\[ \int \int_R f(x, y) \, dA = \int \int_{R_1} f(x, y) \, dA + \int \int_{R_2} f(x, y) \, dA \]

\[ = F_1 \text{area}(R_1) + F_2 \text{area}(R_2) \]

\[ = 2(F_1 + F_2) \]

\[ = 2(8 + \ln 3) \]

where \( F_1, F_2 \) are constants

\[ F_1 = 2 + \ln 1 = 2 \]

\[ F_2 = 6 + \ln 3 \]

\[ R.S. = g(c_1) \text{area}(R_1) + g(c_2) \text{area}(R_2) + \cdots \]

\[ = g(c_1) + g(c_2) + g(c_3) + g(c_4) \]

where \( c_1, c_2, c_3, c_4 \) are the centers of the rectangles

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6. (Double Integrals) Complete one of the following.

(a) [100%] Find the volume of the solid in the first octant bounded by the surface \( z = e^{x-y} \), the plane \( x + y = 1 \) and the coordinate planes.

(b) [100%] Let \( R \) be the planar \( xy \)-region described by \( 0 \leq x \leq 1, \ 0 \leq y \leq \sqrt{1-x^2} \). Let \( f(x, y) = \frac{1}{\sqrt{4 - x^2 - y^2}} \). Evaluate \( \iint_R f(x, y) \, dA \) using polar coordinates.

\[ \text{Vol} = \iint_R e^{x-y} \, dA \]
\[ R = \left\{ (x, y) : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1-x \right\} \]

\[ \begin{align*}
\text{Vol} &= \int_0^1 \int_0^{1-x} e^{x-y} \, dy \, dx \\
&= \int_0^1 \left[ -e^{x-y} \right]_{y=0}^{y=1-x} \, dx \\
&= \int_0^1 (-e^{x-1+x} + e^x) \, dx \\
&= \left[ -\frac{e^{2x-1}}{2} + e^x \right]_0^1 \\
&= -\frac{e}{2} + e + \frac{e-1}{2} \\
&= \frac{e}{2} + \frac{e-1}{2} \\
\end{align*} \]

\[ \begin{align*}
\text{Vol} &= \iint_R \frac{1}{\sqrt{4 - x^2 - y^2}} \, dA \\
R &= \left\{ (r, \theta) : \ 0 \leq \theta \leq \frac{\pi}{2}, \ 0 \leq r \leq 1 \right\} \\
\int_0^1 \int_{\sqrt{4-r^2}}^{\sqrt{1}} r \, dr \, d\theta \\
&= \frac{\pi}{2} \int_0^1 \left[ -\frac{u^{1/2}}{1/2} \right]_{u=4-r^2}^{u=1} \, dr \\
&= \frac{\pi}{2} \left[ \left( \frac{1^{1/2}}{1/2} \right) - \left( \frac{4-r^2}{1/2} \right) \right]_0^1 \\
&= \pi \left( \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} \right) \\
\end{align*} \]

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7. (Surface Area)
Find the area of the part of the conical surface $x^2 + y^2 = z^2$ that is directly above the $xy$-plane triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 4)$. Display all integration steps and include a figure.

$$\text{area} = \int_S ds$$

$$= \int_R \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

$$= \int_0^4 \int_0^{4-x} \sqrt{2} \, dy \, dx$$

$$= \int_0^4 \sqrt{2(4-y)} \, dx$$

$$= -\sqrt{2} \left( \frac{4-x^2}{2} \right) \bigg|_0^4$$

$$= 8\sqrt{2}$$

$z = f(x, y)$

$$= \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + 1}}$$

$R = \text{triangle}$

$R = \int_0^4 \int_0^{4-y} \sqrt{2} \, dx \, dy$

$\frac{\delta x}{\delta y} = \frac{x}{y}$

$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2 + y^2}{2} + 1}$
8. (Triple Integrals) Complete one of the following.

(a) [100%] Evaluate \(\int_{-2}^{2} \int_{x-1}^{x+1} \int_{0}^{\sqrt{2y^2/x}} 3xyz dz dy dx\).

(b) [100%] Find by using triple integration the volume of the solid in the first octant bounded by \(y = 2x^2\) and \(y + 4z = 8\). Draw a figure. Display all steps.

\[
\begin{align*}
V &= \{ (x,y,z) : 0 \leq x \leq 2, 2x^2 \leq y \leq 8, \quad 0 \leq z \leq (8-y)/4 \} \\
\text{vol} &= \iiint dV \\
&= \int_{0}^{2} \int_{2x^2}^{8} \int_{0}^{(8-y)/4} dz dy dx \\
&= \int_{0}^{2} \int_{2x^2}^{8} \left[ \left. z \right|_{0}^{(8-y)/4} \right] dy dx \\
&= \int_{0}^{2} \int_{2x^2}^{8} \left[ \frac{8-y}{4} \right] dy dx \\
&= \int_{0}^{2} \left[ \frac{-y^2}{8} \right]_{2x^2}^{8} dx \\
&= \int_{0}^{2} \left[ \frac{16 - y^2}{8} - \frac{y^2}{8} ight]_{2x^2}^{8} dx \\
&= \int_{0}^{2} \left[ 16 - \frac{25y^2}{8} - 4x^2 + \frac{4y^4}{8} \right] dx \\
&= \int_{0}^{2} \left( 16 - \frac{25y^2}{8} - 4x^2 + \frac{4y^4}{8} \right) dx \\
&= \left[ 16 - \frac{25y^2}{8} - \frac{4x^3}{3} + \frac{32y^4}{10} \right]_{0}^{2} \\
&= \left[ 16 - \frac{32}{3} + \frac{32}{10} \right] \\
&= 156
\end{align*}
\]

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9. (Line Integrals)
(a) [25%] Define work using line integrals. Explain how a line integral equals an ordinary calculus I integral.
(b) [75%] Find the work done by vector force \( \mathbf{F} = (2x - y)\mathbf{i} + (2z)\mathbf{j} + (y - z)\mathbf{k} \) where path \( C \) is the line segment from \((0, 0, 0)\) to \((1, 1, 1)\).

\[ \text{Work done by variable force } \mathbf{F} \text{ along path } C \text{ is} \]
\[ \text{work} = \int_C \mathbf{F} \cdot d\mathbf{r} \]

Here, \( \mathbf{r}(t) \) is a parameterization of curve \( C \), \( a \leq t \leq b \).

In terms of ordinary integrals of calculus I,
\[ \text{work} = \int_a^b (\mathbf{F}(\mathbf{r}(t))) \cdot \mathbf{r}'(t) dt \]

\[ \mathbf{r}(t) = \left( \begin{array}{c} t \\ t \\ t \end{array} \right) \quad 0 \leq t \leq 1 \]

\[ = \int_0^1 \left( \begin{array}{c} 2t \\ 2t \\ 2t \end{array} \right) \cdot \left( \begin{array}{c} t \\ t \\ t \end{array} \right) dt \]
\[ = \int_0^1 (2t)^2 dt \]
\[ = \int_0^1 2t^2 dt \]
\[ = \frac{3}{2} t^2 \bigg|_0^1 \]
\[ = \frac{3}{2} \]

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10. (Divergence and Curl) Complete the following.
   (a) [25%] Define divergence and curl.
   (b) [25%] Compute the divergence of the vector function \( \mathbf{F} = xyz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} \).
   (c) [40%] Compute the curl of the vector function \( \mathbf{F} = (y + z) \mathbf{i} + (x + z) \mathbf{j} + (x + y) \mathbf{k} \).
   (d) [10%] What does divergence measure, in the case of a vector field \( \mathbf{F} \) which represents the velocity field of a fluid?

\[ \text{Div} \left( \mathbf{F} \right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]

\[ \text{Curl} \left( \mathbf{F} \right) = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{array} \right| \]

3 min

\[ \text{Div} \left( \mathbf{F} \right) = yz + 0 + 0 = \left( yz \right) \]

\[ \text{Curl} \left( \mathbf{F} \right) = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & x+y \end{array} \right| \]

\[ = \left( \begin{array}{c} 1-1 \\ -(1-1) \\ 1-1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]

\[ \text{It measures the rate of creation and destruction of fluid.} \]