1996 Utah State Math Contest
University of Utah
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Draft Solution Set

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1996 Junior Exam Grade 7 with Key and Solutions

Answers to 1996 Exam Grade 7			
1. **(a)**	11. **(a)**	21. **(a)**	31. **(a)**
2. **(d)**	12. **(a)**	22. **(a)**	32. **(b)**
3. **(e)**	13. **(d)**	23. **(e)**	33. **(a)**
4. **(a)**	14. **(a)**	24. **(a)**	34. **(d)**
5. **(d)**	15. **(d)**	25. **(c)**	35. **(c)**
6. **(b)**	16. **(c)**	26. **(d)**	36. **(d)**
7. **(b)**	17. **(c)**	27. **(b)**	37. **(e)**
8. **(c)**	18. **(b)**	28. **(a)**	38. **(b)**
9. **(b)**	19. **(c)**	29. **(c)**	39. **(d)**
10. **(c)**	20. **(b)**	30. **(c)**	40. **(a)**

1. **(Defects)** A bolt has probability 3/100 of being defective. How many defective bolts are expected in a package of 1200?

[**a**] 36 (b) 360 (c) 120 (d) 40 (e) 12

Solution to 1. Three out of 100 are expected to be defective. From 1200, the expected number is $(1200/100)(3) = 36$.

2. **(Prices)** Computer X cost \$6000 in 1987 but prices halved every 3 years. Find the cost of X in 1996.

(a) \$1500 (b) \$1250 (c) \$1000 [**d**] \$750 (e) \$500

Solution to 2. The price is $6000(1/2)^{(n-1987)/3}$. For $n = 1996$ the price is $6000(1/2)^3 = 6000/8 = 750$.

3. **(Doubling)** A positive number is multiplied by itself and then the new number is doubled to obtain 32. What is the original number?

(a) 16 (b) 12 (c) 10 (d) 8 [**e**] None

Solution to 3. Let x be the number. Then $2x^2 = 32$. To solve $x^2 = 16$, ask "which number x when squared equals 16", to obtain $x = 4$.

4. **(Sequence)** What is the eighth term divided by the ninth term of the sequence $\frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \frac{4}{5^2}, \ldots$?

[**a**] $\frac{800}{9^3}$ (b) $\frac{8}{900}$ (c) $\frac{700}{8^3}$ (d) $\frac{7}{800}$ (e) None

Solution to 4. The eighth term is $\frac{8}{9^2}$ and the ninth term is $\frac{9}{10^2}$. Their quotient is $\frac{8}{9^2}\frac{10^2}{9} = \frac{800}{9^3}$.

5. **(Cooks)** One Cook makes 50 hamburgers per hour and the other 42. How many can they make together in 3 hours?

(a) 92 (b) 46 (c) 138 [**d**] 276 (e) None

Solution to 5. In one hour they can make $50 + 42 = 92$. In three hours, $(3)(92) = 276$.

6. **(Sound)** Sound travels in air about 344 meters per second. Find the distance in meters to a lightning flash, given the thunder was heard $4\frac{1}{3}$ seconds later, to the nearest integer.

(a) 1460 [**b**] 1491 (c) 1500 (d) 1502 (e) 1516

Solution to 6. The formula $D = RT$ is applied with $R = 344$ and $T = 4\frac{1}{3}$ to give $D = 344\frac{13}{3} = 1490.67$, or about 1491 kilometers.

7. **(Inequalities)** Which is a solution to the inequality $-5 < x < -\frac{11}{9}$?

(a) -5 [**b**] $-\frac{7}{2}$ (c) -1 (d) -6 (e) $-\frac{11}{2}$

Solution to 7. The inequality is strict, which eliminates -5. Both -6 and -1 are out of the interval. So is $-11/2 = -5.5$. On the number line, $-7/2$ is between -5 and $-11/9$, therefore it is the answer.

8. **(Coins)** A glass contains 3 nickels, 6 dimes, 2 quarters and 1 penny. Find the probability of randomly removing a penny or quarter.

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ [**c**] $\frac{1}{4}$ (d) $\frac{1}{3}$ (e) $\frac{3}{4}$

Solution to 8. The sample set consists of 12 items and 3 of these represent success, so the probability is 3/12 or 1/4.

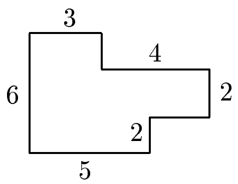
9. **(Shopping)** A shopper in ZCMI Center bought 7 items at \$1.59, 5 at \$1.89 and 8 at \$1.66. Find the average dollar cost per item to the nearest cent.

(a) 1.68 [**b**] 1.69 (c) 1.70 (d) 1.71 (e) 1.72

Solution to 9. The cost is $(7)(1.59) + (5)(1.89) + (8)(1.66) = 33.86$ for $7 + 5 + 8 = 20$ items. The average is $33.86/20 = 1.693$.

10. **(Area)** Find the area of the quadrilateral in the figure. All angles are right angles.

(a) 28 (b) 29 [**c**] 30
(d) 31 (e) 32



Solution to 10. Break the figure into rectangles and apply the area formula: *area* = *length* × *width*.

11. **(Fractions)** If 5/7 of a number is 1200, then what is 7/5 of that number?

[**a**] 2352 (b) 1200 (c) 1680 (d) 857 (e) None

Solution to 11. Let x be the number. Then $(5/7)x = 1200$, giving $x = 1680$. The answer requested is $(7/5)x = (7/5)(1680) = 2352$.

12. (**Divisor**) Let $X = 1337456631234376$. Which number below divides X with zero remainder?

[**a**] 8 (b) 41 (c) 251 (d) 1973 (e) None

Solution to 12. Let $x = 1337456631234376 = 1000(1337456631234) + 376$. Since 1000 is divisible by 8 and 376 is divisible by 8, then so is x. Most graphing calculators in 1996 supply no real help on the problem, due to display limitations and automatic conversions into scientific notation.

13. (**Oranges**) Find the percentage of good oranges in a case of 85 oranges containing seventeen rotten oranges.

(a) 20% (b) 52% (c) 72% [**d**] 80% (e) None

Solution to 13. The percentage of good oranges corresponds to $(85 - 17)/85 = 0.8$ or 80%.

14. (**Inequalities**) If x satisfies $x < 2$ and $2x > -6$, then also

[**a**] $-3 < x < 2$ (b) $-2 < x < 3$ (c) $-2 < x < 2$ (d) $-12 < 2x < 3$ (e) None

Solution to 14. The second inequality can be written as $x > -6/2$ by dividing both sides by 2. Together, the inequalities $x < 2$ and $x > -3$ mean $-3 < x < 2$.

15. (**Systems**) Find the value of y in the linear system $x + 2y = 4$, $y - 3x = 9$.

(a) -2 (b) 1 (c) 2 [**d**] 3 (e) 4

Solution to 15. Solve the first equation for x in terms of y: $x = 4 - 2y$. Put the answer into the second equation $y - 3x = 8$ to get $y - 3(4 - 2y) = 9$. Then $7y = 12 + 9$ and $y = 3$. The other answer is $x = -2$, using the equation $x + 2y = 4$ with value $y = 3$.

16. (**Art Barn**) A gallery 18 by 15 feet by 8 feet high will be redecorated. Find the total square footage of the walls and ceiling.

(a) 399 (b) 528 [**c**] 798 (d) 1056 (e) 1068

Solution to 16. The four walls add $(18)(8) + (18)(8) + (15)(8) + (15)(8)$ square feet. The ceiling is $(18)(15)$ square feet. The total is $528 + 270 = 798$ square feet.

17. (**TV**) A rectangular screen has diagonals of lengths $x + 10$ and $2x - 30$. Find x.

(a) 38 (b) 39 [**c**] 40 (d) 41 (e) 42

Solution to 17. The diagonals are equal: $x + 10 = 2x - 30$. Solve for $x = 40$.

18. (**Quotient**) Evaluate $\dfrac{-(|-13| - |-3|)}{\frac{81}{7} - \frac{22}{14}}$.

(a) -1.6 [**b**] -1 (c) 1 (d) 1.6 (e) None

Solution to 18. The top of the fraction is $|-13| - |-3| = 13 - 3 = 10$. The bottom of the fraction is $81/7 - 22/14 = 81/7 - 11/7 = 10$. Because of the leading minus sign, the answer is $-10/10 = -1$.

19. (**Bagelry**) Bagels priced at \$0.40 were increased in price by $37\frac{1}{2}\%$ and later sold at a 25% off sale. Find the sale price to the nearest cent.

(a) 39 (b) 40 [**c**] 41 (d) 42 (e) 45

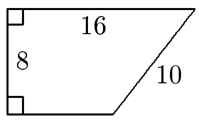
Solution to 19. Use units of cents. The price increase was $(0.375)(40) = 15$ making a new price of $40 + 15 = 55$. The 25% off sale uses a sale price of $(0.75)(55) = 41.25$.

20. (**Altitude**) A triangle has base 40 and area 200. Find the altitude.

(a) 9.5 [**b**] 10 (c) 10.5 (d) 11 (e) 11.5

Solution to 20. The area is $\frac{1}{2}bh$ for base $b = 40$ and unknown altitude h. Given area 200, the equation $300 = \frac{1}{2}(40)h$ is solved for h, giving $h = 10$.

21. (**Trapezoid**) Find the area of the trapezoid.



[**a**] 104 (b) 105 (c) 106 (d) 112 (e) 90

Solution to 21. The rectangular part has area $(8)(10) = 80$. The triangular part has legs x and 8 The hypotenuse is 10. By similarity to a 3-4-5 triangle, $x = (2)(3) = 6$. The total area is $80 + \frac{1}{2}(8)(6) = 104$.

22. (**Powers**) Simplify $(-3x^3)^2 + (2x^2)^3$

[**a**] $17x^6$ (b) $17x^5$ (c) $-x^6$ (d) $-x^5$ (e) $14x^6$

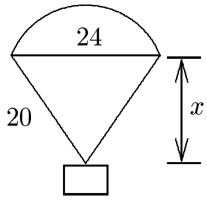
Solution to 22. By exponent rules $(-3x^3)^2 = (-3)^2(x^3)^2 = 9x^6$ and $(2x^2)^3 = (2)^3(x^2)^3 = 8x^6$. The sum is $(9 + 8)x^6$.

23. (**Scores**) Find the *mode* for the scores 7, 9, 12, 18, 4, 16, 8, 21, 13.

(a) 11 (b) 12 (c) 13 (d) 108 [**e**] None

Solution to 23. There is no mode because each score occurs just once.

24. (**Payload**) Supplies on a parachute hang below the canopy of diameter 24ft on ropes 20ft long. Find the distance x in feet from the canopy to the payload.



[**a**] 16 (b) $11\sqrt{2}$ (c) $9\sqrt{3}$ (d) $7\sqrt{5}$ (e) $4\sqrt{11}$

Solution to 24. Let x be the altitude. Then $20^2 = 12^2 + x^2$ by the Pythagorean theorem. The answer is $x = 16$.

25. (**Remainder**) What is the remainder when 7^{1025} is divided by 5?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) 0

Solution to 25. The first few powers of 7 are 7, 49, 343, 2401, 16807. The remainders after division by 5 are 2, 4, 3, 1, 2. The pattern repeats, each four causing the remainders to repeat. Since $1025/4 = 256\frac{1}{4}$, the remainder is the first answer in the list of four repeated values, namely, 2.

26. **(Vertex Angle)** Find the vertex angle of an isosceles triangle with one base angle 40°.

(a) 50° (b) 80° (c) 90° [**d**] 100° (e) None

Solution to 26. The base angles are equal. So both base angles are 40o. The sum of the angles is 180o in a triangle, therefore the vertex angle is $180 - 40 - 40 = 100$.

27. **(Blood Cell)** About how many DNA strands can be placed end–to–end around the equator of a spherical human blood cell of volume $\frac{4}{3}\pi(2^{-17})^3$ cubic meters? Use 2^{-18} meters for the length of a strand.

(a) 11 [**b**] 13 (c) 15 (d) 17 (e) 19

Solution to 27. The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$. The radius R is determined from looking at the expression for the volume: $R = 2^{-17}$. The equator has circumference $C = 2\pi R$. Divide C by $S = 2^{-18}$ to find the number of strands around the equator: $C/S = 2\pi 2^{-17}/2^{-18} = 4\pi = 12.57$.

28. **(Complement)** The measure of an angle is 26° less than four times the measure of its complement. Find the measure of the complement.

[**a**] $23\frac{1}{5}^\circ$ (b) $22\frac{4}{5}^\circ$ (c) $67\frac{1}{5}^\circ$ (d) $66\frac{4}{5}^\circ$ (e) None

Solution to 28. Let x be the complement. Then $90 - x$ is the angle and the text implies $90 - x = 4x - 26$. Solving gives $x = 116/5 = 23.2$.

29. **(Spinners)** Two twelve-hour clock faces are equipped with spinners that always land on a whole number 1 through 12. After spinning each, what is the probability that one or both of the numbers is greater than 8?

(a) $\frac{2}{3}$ (b) $\frac{4}{9}$ [**c**] $\frac{5}{9}$ (d) $\frac{29}{36}$ (e) None

Solution to 29. The spins result in possible events (a, b) where a and b are integers 1–12. Altogether there are $12^2 = 144$ events. An event is successful if $a > 8$ or $b > 8$. These events are enumerated in a neat table and counted to give the success count. The numbers $a = 1$ to $a = 8$ require b to be selected from 9, 10, 11 and 12. Hence there are (8)(4) successful events from this range. For $a = 9$ to $a = 12$, any value of b is allowed, resulting in (4)(12) successful events. The total successful events number $32 + 48 = 80$. The probability is $80/144 = 5/9$.

30. **(Polynomial operations)** Simplify the expression $5(x + 2y)^2 - 2(x^2 - 5y^2)$

(a) $3x^2+10y^2$ (b) $3x^2+30y^2$ [**c**] $3x^2+20xy+30y^2$ (d) $3x^2 + 20xy + 10y^2$ (e) None

Solution to 30. The details: $5(x + 2y)^2 - 2(x^2 - 5y^2) = 5(x^2 + 4xy + 4y^2) - 2x^2 + 10y^2 = 5x^2 + 20xy + 20y^2 - 2x^2 + 10y^2 = 3x^2 + 20xy + 30y^2$.

See *infra* for problems 31-40.

1996 Junior Exam Grade 8 with Key and Solutions

Answers to 1996 Exam Grade 8			
1. **(a)**	11. **(e)**	21. **(b)**	31. **(a)**
2. **(d)**	12. **(e)**	22. **(b)**	32. **(b)**
3. **(a)**	13. **(b)**	23. **(d)**	33. **(a)**
4. **(d)**	14. **(c)**	24. **(c)**	34. **(d)**
5. **(c)**	15. **(c)**	25. **(c)**	35. **(c)**
6. **(d)**	16. **(c)**	26. **(c)**	36. **(d)**
7. **(a)**	17. **(a)**	27. **(d)**	37. **(e)**
8. **(b)**	18. **(a)**	28. **(d)**	38. **(b)**
9. **(e)**	19. **(c)**	29. **(d)**	39. **(d)**
10. **(b)**	20. **(e)**	30. **(d)**	40. **(a)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Probability)** One piece of fruit is drawn randomly from a basket with 2 oranges, 3 bananas and 4 apples. Find the probability that the fruit is not an apple and not a banana.

[**a**] 2/9 (b) 1/3 (c) 4/9 (d) 5/9 (e) 2/3

Solution to 1. There are 2 chances out of 9 to pick an orange.

2. **(Inequalities)** Represent by an inequality: *Cynthia is not more than 16 years old.*

(a) $C < 16$ (b) $C > 16$ (c) $16 \le C$ [**d**] $C \le 16$ (e) None

Solution to 2. It means Cynthia is possibly age 1 through 16, i.e., $C \le 16$.

3. **(Subtraction)** Subtract the sum $[8 + (-7)]$ from the sum $[12 + (-10)]$.

[**a**] 1 (b) −1 (c) −21 (d) −13 (e) 7

Solution to 3. Its $[12 + (-10)] - [8 + (-7)] = 12 - 10 - 8 + 7 = 1$.

4. **(Reading)** Determine the fastest reader from the table below.

(a) Jack (b) Jose (c) Judy [**d**] Jan (e) Jill

Name	Jack	Jose	Judy	Jan	Jill
Words	1222	1345	1100	734	610
Minutes	10	11	9	6	5

Solution to 4. The question reduces to finding the largest fraction among 1222/10, ..., 610/5. With a calculator, the decision is made after computing all decimals. Without a calculator, the final contest is between 1100/9 and 734/6. Multiply by 18 and compare $(2)(1100) = 2200$ with $(3)(734) = 2202$. So Jan is fastest.

5. **(Months)** February is one month before March. What month is 616 months before March?

(a) Sep (b) Oct [**c**] Nov (d) Dec (e) Jan

Solution to 5. The number 616 represents 51 years and 4 months. The question is the same as: *What month is* 4 *months before March*?

6. **(Buttons)** A box with 75 buttons has 24% red ones, $22\frac{2}{3}$% blue ones and the rest are green. What fraction are green?

(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{7}{15}$ [**d**] $\frac{8}{15}$ (e) None

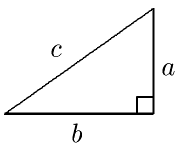
Solution to 6. The number of red ones is $0.24(75) = 18$, and the number of blue ones is $\frac{68}{3}\frac{75}{100} = 17$. The fraction of green ones is $(75 - 18 - 17)/75 = 8/15$.

7. **(Midpoint)** Let P be the midpoint of the segment from $A(8,4)$ to $B(12,12)$. Find the coordinates of P.

[**a**] $(10,8)$ (b) $(8,10)$ (c) $(20,16)$ (d) $(16,20)$ (e) $(4,8)$

Solution to 7. The midpoint (x,y) is given by $x = (8+12)/2$, $y = (4+12)/2$.

8. **(Triangles)** Find the area of the right triangle, given $a = 15$ and $c = 17$.



(a) 59 [**b**] 60 (c) 237/4 (d) 289/4 (e) 239/4

Solution to 8. The Pythagorean Theorem gives the missing leg b in the right triangle from the equation $15^2 + b^2 = 17^2$ or $b = 8$. The area is half the base times the altitude, $(1/2)(15)(8) = 60$.

9. **(Analytic geometry)** Determine the slope of the line through $(-1,8)$ and $(3,-2)$.

(a) 3 (b) -3 (c) $\frac{3}{2}$ (d) $\frac{5}{2}$ [**e**] $-\frac{5}{2}$

Solution to 9. The slope is $m = (-2-8)/(3-(-1)) = -10/4 = -5/2$.

10. **(Trip)** Viktor leaves in his Volvo at 10:00am, driving 55mph. At 11:30am, Jana follows him in her Justy going 50mph. When are they 100 miles apart?

(a) 2:45pm [**b**] 3:00pm (c) 3:15pm (d) 3:30pm (e) 3:45pm

Solution to 10. Jack has traveled $(1.5)(55) = 82.5$ miles when Jill starts her trip. Their speed difference is 5mph. The distance between them grows by 5 miles per hour. When the distance grows by 17.5 miles, then they will be 100 miles apart. This takes $17.5/5 = 3.5$ hours. The time then will be $3\frac{1}{2}$ hours after 11:30am, which is 3:00pm.

11. **(Radicals)** Solve $4 + \sqrt{56 - 15x} = 3x$ for x.

(a) $-\frac{5}{3}$ (b) $-\frac{7}{3}$ (c) $\frac{7}{3}$ (d) 3 [**e**] None

Solution to 11. Put 4 across the equal sign and square both sides to obtain $56 - 15x = 9x^2 - 24x + 16$. Place all terms on the right and factor with the FOIL method into $0 = (3x+5)(3x-8)$. The answer 8/3 is correct. The number $-5/3$ is an *extraneous solution*, it does not satisfy the original equation (the left side is always positive, hence $x > 0$).

12. **(Functions)** Given $f(x) = (x+1)/(x+3)$ and $g(x) = 1/(x-1)$, then find $f(g(1/3))$.

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$ [**e**] $-\frac{1}{3}$

Solution to 12. This is the value of $y = (x+1)/(x+3)$ when $x = g(1/3) = -3/2$. So $y = (-1.5+1)/(-1.5+3) = (-0.5)/(1.5) = -1/3$.

13. **(LCM)** Find the least common multiple of 45, 30, 25.

(a) 11250 [**b**] 450 (c) $\frac{450}{3}$ (d) 100 (e) $\frac{100}{3}$

Solution to 13. The LCM of three numbers is the smallest number N divisible by each of the three numbers. Write $45 = (3)(3)(5)$, $30 = (2)(3)(5)$, $25 = (5)(5)$. Then $LCM = (3)(3)(5)(5)(2) = 450$.

14. **(Jackpot)** A jackpot of \$26.50 is divided with 60¢ to each man, 95¢ to each woman and 70¢ left over. A division of 75¢ per person would be short 50¢. Find the number of women.

(a) 10 (b) 11 [**c**] 12 (d) 13 (e) 14

Solution to 14. Let M be the number of men, W the number of women. Then $75(M+W) = 2650+50$, $60M + 95W = 2650 - 70$. Solving gives $W = 12$ and $M = 24$.

15. **(Radicals)** Simplify $\sqrt{98}\sqrt{75} - \sqrt{14}\sqrt{21} - 4\sqrt{108}\sqrt{2}$.

(a) 9.79 (b) $3\sqrt{6}$ [**c**] $4\sqrt{6}$ (d) $6\sqrt{2}$ (e) $\pi\sqrt{10}$

Solution to 15. Factor the integers as follows: $98 = (2)(7^2)$, $75 = (3)(5^2)$, $14 = (2)(7)$, $21 = (3)(7)$, $108 = (3)(2^2)(3^2)$. Then $\sqrt{6}$ is a common factor and the sum is $(35 - 7 - 24)\sqrt{6}$.

16. **(Rectangle)** A rectangle has diagonal 34. Twice the shorter side x is two more than the longer side. Find x.

(a) 14 (b) 15 [**c**] 16 (d) 17 (e) None

Solution to 16. Let x and $2x - 2$ be the sides of the rectangle. The Pythagorean Theorem applies to give equation $x^2 + (2x-2)^2 = 34^2$. Simplifying, it becomes $5x^2 - 8x - 1152 = 0$ which factors by the FOIL method into $(x-16)(5x+72) = 0$. The answer is $x = 16$.

17. **(Rational expression)** Simplify

$$\frac{x^2-4}{x^2-7x+12}\,\frac{x^2-9}{x^2-2x-8} \div \frac{x^2+x-6}{x^2-6x+8}.$$

[**a**] $\frac{x-2}{x-4}$ (b) $\frac{x-4}{x-2}$ (c) $\frac{x}{x-4}$ (d) $\frac{x-2}{x+4}$ (e) None

Solution to 17. The factorizations are $x^2 - 4 = (x-2)(x+2)$, $x^2 - 7x + 12 = (x-4)(x-3)$, $x^2 - 9 = (x-3)(x+3)$, $x^2 - 2x - 8 = (x-4)(x+2)$, $x^2 + x - 6 = (x-2)(x+3)$, $x^2 - 6x + 8 = (x-4)(x-2)$. After inverting and multiplying, the product reduces to $(x-2)/(x-4)$.

18. **(Quotients)** Find the sum of the solutions to $\dfrac{1}{2x} - \dfrac{9}{x^2+6x} = \dfrac{2-x}{2x+12}$.

[**a**] 1 (b) 4 (c) -1 (d) -3 (e) None

Solution to 18. Clear the fractions to obtain the quadratic $x^2 - x - 12 = 0$ which factors into

$(x-4)(x+3)=0$. The sum of the roots 4 and -3 is 1.

19. **(Auto)** The annual rate of simple interest on a one-year \$7200 loan is 12%. What amount in dollars must be repaid?

(a) 864 (b) 6336 [**c**] 8064 (d) 8117.98 (e) 8113.14

Solution to 19. The simple interest is \$864, computed as $(0.12)(7200)$. The amount is the interest plus the principal, $7200+864=8064$.

20. **(Recording Artists)** In a survey of 900 people concerning two vocalists, 415 liked Paul, 269 liked Ringo and among these, 124 liked both. Estimate the probability that a person dislikes both Paul and Ringo.

(a) 0.21 (b) 0.24 (c) 0.27 (d) 0.30 [**e**] 0.38

Solution to 20. The total who like one or the other or both is 415+269 less the number 124 counted twice, or 560. The total disliking both is not 216, but $900-560=340$. The probability of disliking both is $340/900=17/45=0.38$.

21. **(Powers)** Simplify $\dfrac{(x^3y^{-4})^{-2}}{(x^{-2}y^3)^3}$.

(a) $\frac{1}{x}$ [**b**] $\frac{1}{y}$ (c) $\frac{1}{xy}$ (d) $\frac{1}{y^{12}}$ (e) None

Solution to 21. The numerator is $x^{-6}y^8$ and the denominator is $x^{-6}y^9$. After cancellations of like terms the result is $1/y$.

22. **(Trapezoid)** A trapezoid of area 342 square units has parallel sides of $x-1$ and $2x+7$ units. Given these parallel sides are 12 units apart, find x.

(a) 16 [**b**] 17 (c) 21 (d) 51 (e) None

Solution to 22. The average of the parallel sides $x-1$ and $2x+7$ times the altitude equals the area, which gives the equation $(x-1+2x+7)(12)/2=342$. Solving for x in $(3x+6)(6)=342$ gives $x=51/3=17$.

23. **(Variation)** Let x vary directly as y. Find y at $x=5$, given $y=30$ at $x=3$.

(a) 300 (b) 150 (c) 71 [**d**] 50 (e) 25

Solution to 23. Let $x=ay$. The given $y=30$ at $x=3$ determines constant a by the equation $3=30a$, ie, $a=1/10$. Then $y=x/a$ at $x=5$ gives $y=5/a=5/(1/10)=50$.

24. **(Color–Blind)** Find the probability that two black socks are drawn randomly without replacement from a drawer having 5 black and 8 blue socks.

(a) $\frac{3}{26}$ (b) $\frac{5}{26}$ [**c**] $\frac{5}{39}$ (d) $\frac{1}{39}$ (e) $\frac{14}{39}$

Solution to 24. The events possible are (a,b) where a and b are colors, black or blue. The event size is $13(12)=156$. There are 5(4) events which choose a and b both black. The probability is $20/156=5/39$.

25. **(Remainder)** What is the remainder when 7^{1025} is divided by 5?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) 0

Solution to 25. The first few powers of 7 are 7, 49, 343, 2401, 16807. The remainders after division by 5 are 2, 4, 3, 1, 2. The pattern repeats, each four causing the remainders to repeat. Since $1025/4=256\frac{1}{4}$, the remainder is the first answer in the list of four repeated values, namely, 2.

26. **(Proportions)** Suppose 9 gallons cost \$12.15. Find x, given x pints cost \$2.25.

(a) 12 (b) $\frac{38}{3}$ [**c**] $\frac{40}{3}$ (d) 14 (e) $\frac{20}{3}$

Solution to 26. A gallon is 4 quarts or 8 pints. So $(8)(9)=72$ quarts cost 12.15. The cost per pint is 12.15/72. The proportion is $\dfrac{12.15}{72}=\dfrac{2.25}{x}$. The answer is $x=\dfrac{40}{3}$.

27. **(Distance)** The distance between points $(-a,10)$ and $(-3,5)$ is 13. Find $a>0$.

(a) -9 (b) 9 (c) 12 [**d**] 15 (e) None

Solution to 27. Solve $(-a+3)^2+(10-5)^2=13^2$ for $a>0$. The possible answers for a are 15 and -9, but only one is positive.

28. **(Polynomials)** Find the coefficient of the term x^3 in the polynomial $(x+3)^2(2x-11)(5x-4)$.

(a) 1 (b) 3 (c) -5 [**d**] -3 (e) -1

Solution to 28. Write out the two quadratics $(x+3)^2=x^2+6x+9$ and $(2x-11)(5x-4)=10x^2-63x+44$. The two terms in the product of these quadratics, which contain the term x^3, are $60x^3$ and $-63x^3$. The desired term is $-3x^3$ with coefficient -3. Alternatively, the sum of the roots of the quartic equals $-3-3+\frac{11}{2}+\frac{4}{5}=0.3$, hence the coefficient of x^3 is $(10)(-1)(0.3)=-3$.

29. **(Quotient)** Simplify:

$$\left(1-\frac{1}{1+x}\right)^{-1}+\left(\frac{1}{1-x}-1\right)^{-1}$$

(a) -2 (b) 2 (c) $-2x$ [**d**] $\frac{2}{x}$ (e) None

Solution to 29. The first reciprocal is $\frac{1+x}{x}$. The second is $\frac{1-x}{x}$. Their sum is $2/x$.

30. **(Area)** Find the diameter of a circle of area $\frac{\pi}{25}$.

(a) 2π (b) $\frac{\sqrt{\pi}}{5}$ (c) $\frac{1}{5\sqrt{\pi}}$ [**d**] 0.4 (e) 0.5

Solution to 30. The area is πR^2 in terms of the radius R. The equation $\pi R^2=\pi/25$ is solved for $R=1/5$ and inserted into the diameter formula $D=2R=2/5$.

See *infra* for problems 31-40.

1996 Junior Exam Grade 9 with Key and Solutions

Answers to 1996 Exam Grade 9			
1. **(b)**	11. **(c)**	21. **(c)**	31. **(a)**
2. **(c)**	12. **(a)**	22. **(c)**	32. **(b)**
3. **(b)**	13. **(c)**	23. **(d)**	33. **(a)**
4. **(e)**	14. **(a)**	24. **(c)**	34. **(d)**
5. **(d)**	15. **(b)**	25. **(e)**	35. **(c)**
6. **(d)**	16. **(d)**	26. **(c)**	36. **(d)**
7. **(c)**	17. **(b)**	27. **(b)**	37. **(e)**
8. **(b)**	18. **(d)**	28. **(d)**	38. **(b)**
9. **(a)**	19. **(c)**	29. **(b)**	39. **(d)**
10. **(e)**	20. **(a)**	30. **(a)**	40. **(a)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Pricing)** Two parkas are originally priced at \$150 and \$200. Sales tax is 6%. Find the difference between their register prices at a 30% off sale.

(a) \$15.90 **[b]** \$37.10 (c) \$53 (d) \$47.10 (e) \$27.10

Solution to 1. The 150-dollar parka costs $(0.7)(150)(1.06)$ and the 200-dollar parka costs $(0.7)(200)(1.06)$. The difference is $(0.7)(50)(1.06) = 37.1$.

2. **(Triangle)** A triangle has angles of x, $x+5$ and $2x+3$ degrees. Find the sum of the two smaller angles.

(a) 89 (b) 90 **[c]** 91 (d) 132 (e) 137

Solution to 2. The angles add to 180, giving the equation $x + x + 5 + 2x + 3 = 180$, having solution $x = 43$. The other angles are $(43) + 5 = 48$ and $2(43) + 3 = 89$. The sum of the two smaller angles is $43 + 48 = 91$.

3. **(Equations)** The system $ax - by = 6$, $ax + by = 10$ has the solution $x = -1$, $y = 2$. Find b.

(a) 0 **[b]** 1 (c) -8 (d) -1 (e) 9

Solution to 3. Substitute $x = -1$ and $y = 2$ to obtain the system $-a-2b = 6$, $-a+2b = 10$. Add to obtain $a = -8$. Substitute $a = -8$ into $-a + 2b = 10$ to get $2b = 2$ or $b = 1$.

4. **(Bacteria)** A population of bacteria triples each day, reaching a size of 722925 in 120 hours. Find the initial population size.

(a) 2995 (b) 2990 (c) 2985 (d) 2980 **[e]** 2975

Solution to 4. Let x be the initial population size. Then after n hours the population is $3^{n/24}(x)$. After 120 hours, $n/24 = 120/24 = 5$ and $3^5 x = 722925$. Dividing by 3 five times into 722925 gives 2975.

5. **(Similarity)** A triangle has longest side $x + 30$ and shortest side 8. It is similar to a triangle of longest side x and shortest side 3. Find x.

(a) 48 (b) 30 (c) 20 **[d]** 18 (e) None

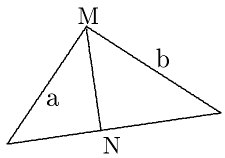
Solution to 5. By similarity, $(x + 30)/x = 8/3$. Then $x = 18$.

6. **(Buttons)** A box with 75 buttons has 24% red ones, $22\frac{2}{3}$% blue ones and the rest are green. What fraction are green?

(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{7}{15}$ **[d]** $\frac{8}{15}$ (e) None

Solution to 6. The number of red ones is $0.24(75) = 18$, and the number of blue ones is $\frac{68}{3}\frac{75}{100} = 17$. The fraction of green ones is $(75 - 18 - 17)/75 = 8/15$.

7. **(Altitude)** In the figure, MN is both a median and an altitude. Given $a = x+7$ and $2b = 3x-15$, find the unknown x.



(a) 18 (b) 21 **[c]** 29 (d) 32 (e) None

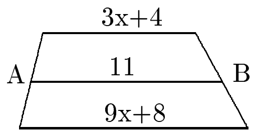
Solution to 7. The triangle must be isosceles, therefore $2a = 2b$ and $3x - 15 = 2(x + 7)$. Solving determines $x = 29$.

8. **(Inequalities)** Describe the x's such that $3(2x - 5) + 5 > -5(1 - x) - 6$ and $2(x + 1) + 1 > 5x - 9$.

(a) $-2 < x < 4$ **[b]** $-1 < x < 4$ (c) $0 < x < 4$ (d) $1 < x < 4$ (e) $2 < x < 4$

Solution to 8. The first reduces to $6x - 14 + 1 > 5x-5-9$ or $x > -1$. The second becomes $2x+2+1 > 5x - 9$ or $12/3 > x$. Together, $4 > x > -1$.

9. **(Trapezoid)** Find x in the trapezoid, given A and B are midpoints.



[a] $\frac{5}{6}$ (b) $\frac{6}{7}$ (c) $\sqrt{\frac{3}{4}}$ (d) $\sqrt{\frac{4}{5}}$ (e) 1

Solution to 9. The average of opposite sides $3x+4$ and $9x + 8$ is 11. Then $6x + 6 = 11$ or $x = 5/6$.

10. **(Rectangle)** A rectangle has a perimeter of 42cm. Which of the following is *definitely not* its area in cm^2?

(a) 107 (b) 108 (c) 109 (d) 110 **[e]** 111

Solution to 10. Let a and b be its dimensions. Then $2a + 2b = 42$, making $a + b = 21$. The area is $ab = a(21 - a)$. The largest positive value of the quadratic $y = x(21 - x)$ occurs at its vertex, at $x = 21/2$, $y = 441/4$. No rectangle can have area larger than 441/4 and perimeter 42. This eliminates all but one of the answers.

11. **(Complement)** The measure of an angle is 2/3 the measure of its supplement. Find the measure of its complement.

(a) 12° (b) 16° **[c]** 18° (d) 36° (e) None

Solution to 11. Let x be the angle. The supplement is $180 - x$. Then $x = 2(180 - x)/3$ gives $5x = 360$ and $x = 72$. The complement is $90 - x$ or 18.

12. (**Trapezoid**) Find the length of the median of a trapezoid with vertices $(0,0)$, $(0,3)$, $(4,5)$ and $(4,-2)$.

[**a**] 5 (b) 4 (c) 3 (d) 2 (e) None

Solution to 12. The parallel bases have lengths 3 and 7, by the distance formula. The median length is half of their sum, $(3+7)/2 = 5$. It can also be worked using the midpoint formula and a suitable diagram.

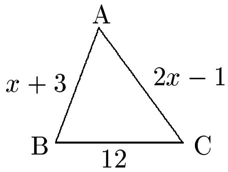
13. (**Line**) Determine the value of r such that the line through the points $(4,r)$ and $(r,2)$ is perpendicular to $5y - 3x = 1$.

(a) -7 (b) 1 [**c**] 7 (d) -1 (e) None

Solution to 13. The slope of $5y - 3x = 1$ is $3/5$. A perpendicular line has slope equal to the negative reciprocal of $3/5$, that is, $m = -5/3$. The slope formula $m = (y_2 - y_1)/(x_2 - x_1)$ applies to give the equation $-5/3 = (2-r)/(r-4)$. Clearing fractions gives $-5(r-4) = 3(2-r)$ which can be solved for $r = 7$.

14. (**Vertex**) Triangle ABC is isosceles with perimeter between 23 and 33. Locate the vertex angle.

[**a**] $\angle A$ (b) $\angle B$ (c) $\angle C$
(d) undecidable (e) None



Solution to 14. The inequality $23 < x + 3 + 2x - 1 + 12 < 33$ means $3 < x < \frac{19}{3}$. To be isosceles, one of these must be true: (1) $x+3 = 12$, (2) $2x-1 = 12$ or (3) $2x - 1 = x + 3$. The only one that can work is (3), which reduces to $x = 4$. So $AB = AC = 7$ and the angle at A is the vertex.

15. (**Inequality**) Solve $|x-2| + |x+2|^2 \le 4$.

(a) $|2x+3| \le 1$
[**b**] $|2x+5| \le \sqrt{33}$ (c) $|x-2| \le 2$ (d) $|x+2| \le 2$
(e) $|x+2| \le 1$

Solution to 15. Since $|x-2| = \pm(x-2)$, the left side of the inequality stands for $(x-2) + (x-2)^2$ or $-(x-2)+(x-2)^2$. Expanding results in two quadratic inequalities, $x^2 + 5x - 2 \le 0$ and $x^2 + 3x + 2 \le 0$. Add $(5/2)^2$ and $(3/2)^2$, respectively, across these inequalities in order to complete the square on each. Taking roots gives $|x + 5/2| \le \sqrt{33}/2$ and $|x + 3/2| \le 1/2$. The latter is $-2 \le x \le -1$, which happens to be included in the first, which is about $-5.37 \le x \le .37$. Therefore, the two inequalities can be combined into $|2x+5| \le \sqrt{33}$.

16. (**Functions**) Given $f(x) = \dfrac{x^2 - x + 2}{8x^2 - 5x + 16}$, find $f(2+\sqrt{2})$.

(a) $\frac{1}{4}$ (b) 9 (c) $\frac{9}{4}$ [**d**] 3^{-2} (e) 5^{-2}

Solution to 16. This is the value of $y = (x^2 - x + 2)/(8x^2 - 5x + 16)$ when $x = 2 + \sqrt{2}$. Then $x^2 - x + 2 = 4 + 4\sqrt{2} + 2 - 2 - \sqrt{2} + 2 = 6 + 3\sqrt{2}$ and $8x^2 - 5x + 16 = 8(4 + 4\sqrt{2} + 2) - 10 - 5\sqrt{2} + 16 = 54 + 27\sqrt{2}$, therefore $y = 1/9$ which is 3^{-2}.

17. (**Volume**) A cube has surface area $25\sqrt{2}$ cm^2. Find its volume in cm^3.

(a) $\frac{12\sqrt{5}}{6\sqrt{2}}$ [**b**] $\frac{125\sqrt[4]{2}}{6\sqrt{3}}$ (c) $\frac{43}{3}$ (d) 14.3 (e) None

Solution to 17. The surface area of a cube of side x is $6x^2$. The equation $6x^2 = 25\sqrt{2}$ determines $x = \frac{5\sqrt[4]{2}}{\sqrt{6}}$. The volume is $V = x^3 = \frac{5^3\sqrt{2}\sqrt[4]{2}}{6\sqrt{6}}$.

18. (**Speedboat**) A powerboat goes upriver 30 miles in 300 minutes. It returns downriver in 3 hours. Find the river's speed in mi/hr.

(a) 5 (b) 4.95 (c) 3 [**d**] 2 (e) 1

Solution to 18. Let r be the speed of the river and d the speed of the boat in still water. Then $(d-r)(5) = 30$ and $(d+r)(3) = 30$. These equations reduce to $d - r = 6$ and $d + r = 10$. Adding gives $2d = 16$ or $d = 8$. Then $r = 10 - d = 2$.

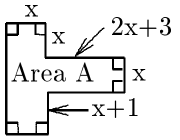
19. (**Color–Blind**) Find the probability that two black socks are drawn randomly without replacement from a drawer having 5 black and 8 blue socks.

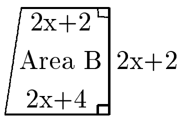
(a) $\frac{3}{26}$ (b) $\frac{5}{26}$ [**c**] $\frac{5}{39}$ (d) $\frac{1}{39}$ (e) $\frac{14}{39}$

Solution to 19. The events possible are (a,b) where a and b are colors, black or blue. The event size is $13(12) = 156$. There are $5(4)$ events which choose a and b both black. The probability is $20/156 = 5/39$.

20. (**Quadratics**) Find the smaller of the Given the T-shaped area A and the trapezoidal area B below, find the smaller area, assuming $0 < x < 2$.

[**a**] $5x^2 + 4x$ (b) $4x^2 + 6x$ (c) $(2x+2)^2$
(d) $4x^2 + 4x$ (e) $-x^2 + 6x + 6$





Solution to 20. Let a be the sum of the areas of the two rectangles in area A. Then $a = 5x^2 + 4x$. Let b be the area of the triangle plus the square in area qB. Then $b = (2x+2)^2 + (2x+2)$ and $b - a = -x^2 + 6x + 6 > -4 + 6x + 6 > 0$ (because $0 < x < 2$ implies $-x^2 > -4$). The inequality $b - a > 0$ means $b > a$ and a is the smallest.

21. (**Radicals**) Find the sum of the roots to $\sqrt{5x^2 + 10x - 11} = x + 1$.

(a) -3 (b) -2 [**c**] 1 (d) 2 (e) 3

Solution to 21. The equation squared is $5x^2 + 10x - 11 = x^2 + 2x + 1$, which simplifies into $4x^2 + 8x - 12 = 0$, having roots 1 and -3. But -3 is not a root of the original equation. So the sum of the roots is 1.

22. (**Remainder**) What is the remainder when 7^{1025} is divided by 5?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) 0

Solution to 22. The first few powers of 7 are 7, 49, 343, 2401, 16807. The remainders after division by 5 are 2, 4, 3, 1, 2. The pattern repeats, each four causing the remainders to repeat. Since $1025/4 = 256\frac{1}{4}$, the remainder is the first answer in the list of four repeated values, namely, 2.

23. **(Functions)** Given $f(t) = 19|t+9| - |9-t|$ and $g(t) = 10 + t$, evaluate $f(g(-19))$.

(a) 18 (b) 19 (c) -19 [**d**] -18 (e) 0

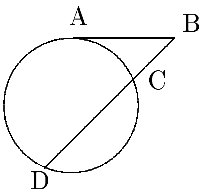
Solution to 23. First, $g(-19) = -9$. Then $f(g(-19)) = f(-9) = 19|-9+9| - |9-(-9)| = -18$.

24. **(Short Side)** Two opposite sides of a parallelogram are $4x+5y$ and $10x+14$. The other sides are $x+5y$ and $y+17$. Find the shortest side.

(a) 4 (b) 20 [**c**] 21 (d) 24 (e) None

Solution to 24. Opposite sides of a parallelogram are equal. The equations are $4x+5y = 10x+14$ and $x+5y = y+17$. Simplifying, they become $-6x+5y = 14$ and $x+4y = 17$. The solution is $x = 1$, $y = 4$. Then the sides of the parallelogram are 21 and 24.

25. **(Tangents)** Given segment $\overline{AB}$ tangent at A, $AB = 9x-22$, $BC = 7x-18$ and $DC = 2x$, find the sum of all possible solutions x.



(a) $\frac{8}{3}$ (b) 4 (c) $\frac{16}{3}$ (d) 6 [**e**] None

Solution to 25. **Theorem**: *If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment is equal to the product of the measures of the secant segment and its external secant segment.* See Merrill Geometry (1993), p 449. Applying the theorem gives $(9x-22)^2 = (7x-18)(9x-18)$. Taking roots gives $x = 10/3$ and $x = 8/3$. Neither root is extraneous. Therefore the sum of the roots is 6.

26. **(Median)** Let $\triangle ABC$ have sides 12, 16 and 20. Find the length of the longest median to the nearest tenth.

(a) 14.1 (b) 14.4 [**c**] 17.1 (d) 17.3 (e) 10.0

Solution to 26. Assume the triangle has vertices $(0,0)$, $(12,0)$ and $(12,16)$. Using the midpoint formula, the missing endpoint of each median can be computed. By geometry, the shortest one connects $(0,0)$ to the midpoint of the hypotenuse, which is at $(6,8)$. The length of that median is 10, by the distance formula. The other medians have lengths $\sqrt{208} = 14.42$ and $\sqrt{292} = 17.09$, by application of the same method.

27. **(Fractions)** Find the perimeter of a quadrilateral with sides $\frac{1}{x-2}$, $\frac{x}{x-2}$, $\frac{1}{x+2}$ and $\frac{x}{x+2}$, given $x = 1+\sqrt{3}$.

(a) $\frac{7}{2}\sqrt{3}$ [**b**] $3+\frac{5}{\sqrt{3}}$ (c) $4+\sqrt{3}$ (d) $5+\frac{1}{\sqrt{3}}$ (e) None

Solution to 27. Add the four fractions to get $\frac{2x(1+x)}{x^2-4}$. Set $x = 1+\sqrt{3}$ to get $3+\frac{5}{\sqrt{3}}$.

28. **(Area)** A circle and a square of the same area have perimeters A and B, respectively. Find A/B.

(a) π (b) $\frac{\pi}{2}$ (c) $\sqrt{\pi}$ [**d**] $\frac{\sqrt{\pi}}{2}$ (e) None

Solution to 28. Let the circle have radius R. Let the square have side x. Then $A = 2\pi R$ and $B = 4x$. Given equal areas, $\pi R^2 = x^2$. Then $x = \sqrt{\pi}R$ and the ratio is $A/B = 2\pi R/(4x) = \pi R/(2\sqrt{\pi}R) = \sqrt{\pi}/2$.

29. **(Time)** Find the *exact* time between 7 and 8 when the minute hand of a watch is precisely 10 minutes in advance of the hour hand.

(a) 7:49 [**b**] $7{:}49\frac{1}{11}$ (c) $7{:}49\frac{2}{11}$ (d) $7{:}49\frac{3}{11}$ (e) $7{:}50\frac{1}{11}$

Solution to 29. Let x be in minutes. The hour hand is at angle $H = 210+30(x/60)$, the minute hand is at angle $M = 360(x/60)$ and the clock position advance for 10 minutes on the dial is 60. The equation to determine x is $M - H = 60$, which becomes $360(x/60) - 210 - 30(x/60) = 60$. Then $x = 540/11$, giving the answer $49\frac{1}{11}$ minutes after 7:00.

30. **(Variation)** Let z vary directly with y^2 and inversely with x, taking on the value $z = a^2+1$ when $y = 2/3$ and $x = 4/(a^2+1)$. Find x when $z = 3(a^2+1)$ and $y = a^2+1$.

[**a**] $3a^2+3$ (b) a^2+1 (c) $8a^2+8$ (d) $9a^2+9$ (e) None

Solution to 30. Let $z = cy^2/x$ or equivalently $xz = cy^2$. Then $y = 2/3$ and $x = 4/(a^2+1)$ at $z = a^2+1$ gives a value for c, due to the equation $(4/(a^2+1))a^2+1 = c(2/3)^2$. Solving for c produces $c = 9$. To find x, solve $3(a^2+1)x = 9(a^2+1)^2$ for $x = 3a^2+3$.

Problems 31–40 have a higher weight in the Junior competition between grades 7, 8 and 9.

31. **(Counting)** How many integers between 100 and 200 contain either the digit 5 or the digit 7?

[**a**] 36 (b) 37 (c) 38 (d) 39 (e) 40

Solution to 31. The numbers of the form $1d5$ and $1d7$ for d equal to 0, 1, 2, 3, 4, 6, 8, 9 account for 16 numbers. The numbers of the form $15d$ and $17d$ for $0 \le d \le 9$ account for 20 numbers. The total is $16+20 = 36$.

32. **(Cards)** Four cards are face down on the table. Two are Aces and two are Jacks. Pick two and turn them over. Find the probability of getting an Ace and a Jack.

(a) $\frac{3}{4}$ [**b**] $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{8}$ (e) None

Solution to 32. There are 12 outcomes and $4\cdot 2$ are favorable. The probability is $8/12$.

33. **(Jackpot)** A jackpot of \$47 is divided into piles A, B, C, D so that A and B together have \$27, A and C have \$25 and finally A and D have \$23. Find the number of dollars in D.

[a] 9 (b) 10 (c) 14 (d) 16 (e) 17

Solution to 33. There are four equations in four unknowns, $A+B+C+D=47$, $A+B=27$, $A+C=25$, $A+D=23$. Add the last three to obtain $3A+B+C+D=75$. Subtract the first from it to get $2A=28$ and $A=14$. Then $D=23-A=9$.

34. **(Composites)** Let $X = 300,100,200,517,201,275$. One number below is a factor of X. Find it.

(a) 21 (b) 33 (c) 69 **[d]** 75 (e) 100

Solution to 34. Write $X=100Y+75$. Then $X=25(4Y+3)$, so 25 divides X. The sum of the digits of X is $3+1+2+5+1+7+2+1+2+7+5=36$ which is divisible by 3, therefore 3 divides X. This shows $(3)(25)$ divides X.

35. **(Game Wheel)** A Wheel is labeled, clockwise along its circumference, with equally spaced numerals 1, 2, ..., 21. A stationary marker points to 1. After 62 seconds of counter–clockwise rotation, the marker is at 8. If numeral 21 passed the marker three times, then find the marker numeral after only 31 seconds.

(a) 13 (b) 14 **[c]** 15 (d) 16 (e) 17

Solution to 35. The wheel completes $3\frac{7}{21}$ rotations in 62 seconds, hence half that many in 31 seconds, which is $1\frac{14}{21}$. The marker points at numeral 15.

36. **(Fractions)** Simplify $\frac{1+3x}{1-2x}$ for $x=\frac{a-2}{2a+1}$ where $a>3$.

(a) $\frac{a-2}{a+2}$ (b) $\frac{a-1}{a+1}$ (c) $2a-1$ **[d]** $a-1$
(e) a^2-1

Solution to 36. Given $x=(a-2)/(2a+1)$, then the fraction $\frac{1+3x}{1-2x}$ becomes a complex fraction, in which multiplication by $\frac{2a+1}{2a+1}$ simplifies the result to $\frac{2a+1+3a-6}{2a+1-2a+4}$. The latter reduces to $a-1$.

37. **(Odometer)** An auto odometer displays miles from 0 to 99999. In the first 1000 miles, how many displays have sum of the digits greater than 20?

(a) 42 (b) 80 (c) 82 (d) 83 **[e]** 84

Solution to 37. The first display is 399. The next ones are in groups starting at 489, 579, 669, 759, 849 and 939. The group sizes are $n(n+1)/2$ for $n=1,\ldots,7$. The total is $1+3+6+10+15+21+28=84$.

38. **(Pentagon)** A pentagon with sides 3, 2, 3, 2 and 5/2 happens to have an inscribed circle of diameter 13/4. Find the area of the pentagon.

(a) $\frac{325}{16}$ **[b]** $\frac{325}{32}$ (c) $\frac{325}{64}$ (d) $\frac{163}{16}$ (e) $\frac{81}{8}$

Solution to 38. The pentagon is the sum of five triangles of altitude $h=13/8$. The area is $3\cdot h/2+2\cdot h/2+3\cdot h/2+2\cdot h/2+(5/2)\cdot h/2 = (25/2)(h/2)=325/32$.

39. **(Ramp)** Find the slope of a ramp which raises an object 1m vertically for each 5m of travel on the ramp surface.

(a) $\frac{1}{5}$ (b) $\frac{4}{\sqrt{6}}$ (c) $\frac{5}{\sqrt{6}}$ **[d]** $\frac{\sqrt{6}}{12}$ (e) None

Solution to 39. Form a right triangle with hypotenuse 5 and legs 1, x. Then $x^2+1^2=5^2$, giving $x=2\sqrt{6}$. The slope is *rise/run*, which is $1/x=1/(2\sqrt{6})=\sqrt{6}/12$.

40. **(Missile)** A missile hits its target with probability 0.3. Find the least number of missiles to be launched to insure an 80% probability of hitting a target.

[a] 5 (b) 6 (c) 8 (d) 9 (e) None

Solution to 40. The missile misses with probability 0.7, so n missiles miss the target with probability 0.7^n. To find the least n, solve the inequality $1-0.7^n>0.8$ by trial–and–error. The least one is $n=5$.

1996 Senior Exam Grade 10 with Key and Solutions

Answers to 1996 Exam Grade 10			
1. **(b)**	11. **(e)**	21. **(d)**	31. **(c)**
2. **(c)**	12. **(c)**	22. **(c)**	32. **(a)**
3. **(e)**	13. **(d)**	23. **(e)**	33. **(a)**
4. **(d)**	14. **(c)**	24. **(a)**	34. **(e)**
5. **(d)**	15. **(d)**	25. **(a)**	35. **(a)**
6. **(d)**	16. **(a)**	26. **(c)**	36. **(a)**
7. **(c)**	17. **(a)**	27. **(a)**	37. **(b)**
8. **(b)**	18. **(c)**	28. **(a)**	38. **(d)**
9. **(b)**	19. **(d)**	29. **(d)**	39. **(c)**
10. **(d)**	20. **(e)**	30. **(e)**	40. **(d)**

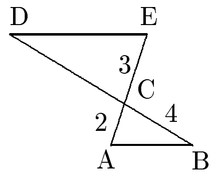
Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Ratio)** The perimeter of a triangle is 66 units and the sides have measures in the ratios 2:4:5. Find the number of units in the shortest side.

(a) 6 **[b]** 12 (c) 18 (d) 24 (e) 30

Solution to 1. The measures of the sides can be written as $2x$, $4x$ and $5x$. The perimeter is $2x+4x+5x=66$. Then $x=6$, but x is not a side; the sides are $2x=12$, $4x=24$ and $5x=30$.

2. **(Geometry)** In the diagram, let segments $\overline{AB}$ and $\overline{DE}$ be parallel and let $AC=2$, $BC=4$ and $CE=3$. Find CD.



(a) 4 (b) 5 **[c]** 6 (d) 7
(e) None

Solution to 2. Let $x=CD$. The triangles ABC and EDC are similar, therefore $\frac{3}{2}=\frac{x}{4}$, which can be solved for $x=6$.

3. **(Midway)** The line $8x + 5y = c$ is midway between and parallel to the parallel lines $8x + 5y = 3$ and $8x + 5y = 5$. Which is a possible value of c?

(a) 5 (b) 6 (c) 7 (d) 8 [**e**] None

Solution to 3. Graph the lines. The "midway" requirement means $\frac{1}{2}(3 + 5) = c$, giving $c = 4$.

4. **(Midpoint)** Given $A(x, -2x + 1)$ and $B(-x, 2x + 1)$, find the midpoint of segment $\overline{AB}$.

(a) (x, x) (b) $(0, 0)$ (c) $(-1, 0)$ [**d**] $(0, 1)$ (e) None

Solution to 4. The midpoint is $((x-x)/2, (-2x+1+2x+1)/2) = (0, 1)$.

5. **(Quadrilateral)** Find x, given a quadrilateral has exterior angles 65, 75, x and $3x$ (in degrees).

(a) 49° (b) 51° (c) 53° [**d**] 55° (e) None

Solution to 5. The sum of the exterior angles must be 360°. This produces the equation $360 = 65 + 75 + x + 3x$ for the unknown x. Solving, $x = 55$.

6. **(Coolant)** Mix 2, 5 and 7 liters of coolant at temperatures 50°, 67° and 73° , respectively, to create a mixture at x degrees. Find x to the nearest degree.

(a) 65 (b) 66 (c) 67 [**d**] 68 (e) 70

Solution to 6. The mixture equation is $(2 + 5 + 7)x = 2(50) + 5(67) + 7(73)$. The answer is $x = 946/14 = 67.57$. The closest answer is 68.

7. **(Composition)** Simplify $(a^2 + 5ab + 6b^2)/(a + 3b)$ and evaluate at $a = 3x^2 - 1$, $b = 1 - x^2$.

(a) $2x^2$ (b) $3x^2 - 1$ [**c**] $1 + x^2$ (d) $x^2 - 1$ (e) x^2

Solution to 7. Factor $a^2 + 5ab + 6b^2 = (a + 3b)(a + 2b)$, cancel $(a + 3b)$ to get $(a + 2b)$. Substitute for a and b to obtain $1 + x^2$.

8. **(Tables)** Find an entry $x > 0$ that completes the table, given $y = \dfrac{x}{x+4}$.

x	0	1	
$3xy$	0	$\frac{3}{5}$	$\frac{1}{13}$

(a) 1/5 [**b**] 1/3 (c) 2/5 (d) 3/5 (e) 2/3

Solution to 8. Solve $3x^2/(x+4) = 1/13$ for x and compare to the table to select 1/3. Alternatively, try each answer in the formula $3xy$.

9. **(Interest)** An investment at 10% interest compounded annually earns \$60.44 in 5 years. Approximate the interest in dollars at six years.

(a) 74 [**b**] 76 (c) 79 (d) 81 (e) 83

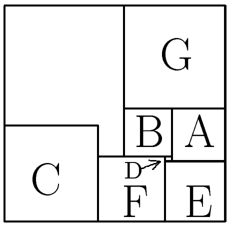
Solution to 9. The equation is $x(1.1)^5 - x = 60.44$. Then $x(1.1)^6 - x = 76.38$.

10. **(Mixture)** Add 50 mL of water to 125 mL of a 20% salt solution. What percentage of the new solution is salt?

(a) 1/7 (b) 14 (c) $14\frac{1}{7}$ [**d**] $14\frac{2}{7}$ (e) $14\frac{3}{7}$

Solution to 10. The amount of salt is $(0.2)(125)$. The amount of solution is $50 + 125 = 150$. The fraction of salt is $(0.2)(125)/175$ which is $14\frac{2}{7}$ percent.

11. **(Squares)** In the big square figure, assume shapes A through G are also squares. If A has area 64 cm^2 and B has diagonal $7\sqrt{2}$ cm, then find the side of C.



(a) 8 (b) 9 (c) 11 (d) 12 [**e**] 13

Solution to 11. Squares A and B have sides 8 and 7, respectively. Square D has side 1. Then square E has side 9 and to its left square F has side 10. Square G has side $7 + 8 = 15$, due to squares A and B filling a side. Then the main square has side $15+8+9 = 32$. Finally, square C has side $32 - 10 - 9 = 13$.

12. **(Population)** In 1990, the population of a city was 14344, a decrease of $8\frac{1}{3}\%$ from its 1980 population. The 1980 population was an increase of $8\frac{2}{3}\%$ from its 1970 population. Find the 1970 population.

(a) 14298 (b) 14392 [**c**] 14400 (d) 14448 (e) None

Solution to 12. Let P and Q be the populations in 1970 and 1980, respectively. Then $8\frac{1}{3}\%$ is 1/12 and $8\frac{2}{3}\%$ is 13/150, giving $14344 = Q - Q/12$ and $Q = P + 13P/150$. Then $P = \frac{150}{163}Q = \frac{150}{163}\frac{12}{11}14344 = 14400$.

13. **(Perfect Cubes)** What fraction of the positive odd integers less than 2199 are perfect cubes?

(a) 1/100 (b) 11/2200 (c) 13/1100 [**d**] 7/1100 (e) 7/2200

Solution to 13. The odd perfect cubes are 1^3, 3^3, …, 13^3. The odd numbers are 1, 3, 5, …, 2199. The counts are 7 and 1100 for fraction 7/1100.

14. **(Symmetry)** How many lines of symmetry are there for the region inside an ellipse of eccentricity 1/2?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) 0

Solution to 14. A line of symmetry is a line that can be drawn through the figure so that it can be folded along the line and the two halves match exactly. The lines of symmetry of an ellipse are just the major and minor axes.

15. **(Polygon)** The sum of the measures of the interior angles in a polygon is 25020. Find the number of sides.

(a) 138 (b) 139 (c) 140 [**d**] 141 (e) 142

Solution to 15. The equation for the number n of sides is $180(n - 2) = 25020$. Solving gives $n = 141$.

16. **(Socks)** Find the probability that two red socks are drawn randomly without replacement from a drawer having 8 red and 5 blue socks.

[**a**] $\frac{14}{39}$ (b) $\frac{10}{39}$ (c) $\frac{5}{39}$ (d) $\frac{1}{39}$ (e) $\frac{56}{169}$

Solution to 16. The events possible are (a, b) where a and b are colors, red or blue. The event size is $13(12) = 156$. There are $8(7)$ events which choose a and b both red. The probability is $56/156 = 14/39$.

17. **(Tangent)** Find the equation of the line through $(3, 4)$ and tangent to the circle of radius 5 and center $(0, 0)$.

[**a**] $3x+4y = 25$ (b) $4x+3y = 24$ (c) $4y-3x = 7$ (d) $4x - 3y = 0$ (e) None

Solution to 17. The equation is $y - 4 = m(x - 3)$. The point $(3, 4)$ is on the circle, therefore m is the negative reciprocal of the slope of the radial line through $(3, 4)$. That line has slope $4/3$, which implies $m = -3/4$. The equation simplifies to $3x + 4y = 25$.

18. **(Water Tank)** A tank has a drain rate of 10% of capacity per hour. Its fill rate is 15% of capacity per hour. When 2/3 full and draining, about how many hours does it take to fill?

(a) $6\frac{1}{3}$ (b) $6\frac{4}{9}$ [**c**] $6\frac{2}{3}$ (d) $5\frac{2}{3}$ (e) $5\frac{1}{3}$

Solution to 18. The fill time t satisfies $t(0.15x - 0.1x) + 2x/3 = x$ where x is the tank capacity. Cancel x and solve for $t = 20/3$.

19. **(Monopoly)** Player A tosses the die and gets number a (1 to 6 possible). Player B tosses and gets number b. Find the probability that $2b > a$.

(a) 23/36 (b) 5/8 (c) 9/10 [**d**] 3/4 (e) 7/8

Solution to 19. There are 36 states $[b, a]$. Possibilities for success are the states $[1, 1]$, $[2, 1]$, $[2, 2]$, $[2, 3]$, $[3, 1]$, $[3, 2]$, $[3, 3]$, $[3, 4]$, $[3, 5]$, $[4, 1]$, $[4, 2]$, $[4, 3]$, $[4, 4]$, $[4, 5]$, $[4, 6]$, $[5, 1]$, $[5, 2]$, $[5, 3]$, $[5, 4]$, $[5, 5]$, $[5, 6]$, $[6, 1]$, $[6, 2]$, $[6, 3]$, $[6, 4]$, $[6, 5]$, $[6, 6]$. The success count is 27. The probability is $27/36$.

20. **(Inequalities)** Find the maximum of $2x + 4y$ subject to the inequalities $x \geq 0$, $y \geq 0$, $x + y \leq 3$ and $3x + y \leq 6$.

(a) 0 (b) 4 (c) 14 (d) 9 [**e**] 12

Solution to 20. The function $z = 2x + 4y$ is a plane in R^3. The inequalities describe a polygon in the plane. The maximum occurs at a vertex of the polygon: $(0, 0)$, $(2, 0)$, $(3/2, 3/2)$, $(0, 3)$. The values of z are 0, 4, 9, 12, respectively. The maximum of $2x + 4y$ is 12.

21. **(Statistics)** Find the average of the mode and the mean for the data set 49, 54, 61, 49, 54, 49, 64 and 60.

(a) 49 (b) 50 (c) 51 [**d**] 52 (e) 55

Solution to 21. The mode is the most frequently listed value: $\overline{m} = 49$. The mean is the average $\overline{x} = 55$. Then $\frac{1}{2}(\overline{m} + \overline{x}) = \frac{1}{2}(49 + 55) = 52$

22. **(Magnitude)** Given $a^3 > b > 1$, which number is largest?

(a) $(a^2b^2)^{101}$ (b) a^{600} [**c**] $(a^2)^{408}$ (d) $(a^4b^4)^{51}$ (e) $(ab)^{203}$

Solution to 22. Change each to powers of a and b. Cancel the common factor of a^{202} and apply comparisons to reduce the contest to $(a^2)^{408}$ and $(a^4b^4)^{51}$. Dividing these two gives $(a^3/b)^{204}$. Then $a^3 > b$ implies the quotient is greater than one, hence $(a^2)^{408}$ is largest.

23. **(Disk)** Select from the list below a point which is inside the circle of center $(-1, 2)$ and radius 3 and outside the circle of center $(-1, 1)$ and radius 2.

(a) $(-2, 1)$ (b) $(-1 - \sqrt{2}, 1)$ (c) $(-1 - \sqrt{3}, 1)$ (d) $(-3, 1)$ [**e**] $(-1 - \sqrt{5}, 1)$

Solution to 23. The inequalities are $(x + 1)^2 + (y - 2)^2 < 9$ and $(x + 1)^2 + (y - 1)^2 > 4$. There is little to do except try the points.

24. **(Sector)** Find the area of a sector of arc length $\pi/6$ and radius $1/\pi$.

[**a**] $\frac{1}{12}$ (b) $\frac{\pi}{36}$ (c) $\frac{11}{126}$ (d) $\frac{\pi}{48}$ (e) None

Solution to 24. The area is a fraction of the area of the whole circle. The fraction is the arc length s of the sector divided by the arc length of the circle: $s/(2\pi r)$. The whole circle area is πr^2. The sector area is $\pi r^2 \frac{s}{2\pi r} = \frac{sr}{2}$. The answer is therefore $\frac{1}{2}\frac{\pi}{6}\frac{1}{\pi} = \frac{1}{12}$.

25. **(Tiles)** Rectangular tiles of width 1 and lengths 1, 2, 3 and 5 touch each other to form a figure. Let x and y be the maximum and minimum perimeters among all figures. Find x/y.

[**a**] $\frac{15}{8}$ (b) 2 (c) $\frac{17}{8}$ (d) $\frac{9}{4}$ (e) $\frac{19}{9}$

Solution to 25. The maximum is 30 obtained by the tiles touching only at a vertex. The minimum is 16 obtained by packing the tiles into a rectangle (various geometries are possible).

26. **(Polynomials)** Let f and g be polynomials of degrees 2 and 6, respectively. Find the degree of the quotient $g(x^2)/f(x^3)$, ignoring any remainder.

(a) 2 (b) 4 [**c**] 6 (d) 7 (e) 12

Solution to 26. The degree of $f(x^3)$ is 6 and the degree of $g(x^2)$ is 12. The quotient has degree 6.

27. **(Trigonometry)** A triangle has sides of length 4, 5 and 8.5 units. Find the cosine of the largest angle.

[**a**] $-\frac{25}{32}$ (b) $-\frac{3}{4}$ (c) $-\frac{13}{16}$ (d) $-\frac{27}{32}$ (e) None

Solution to 27. The largest angle is opposite the longest side. Apply the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$ to obtain $\cos C = (a^2 + b^2 - c^2)/(2ab)$. Then $\cos C = -25/32$.

28. **(Guessing)** Elena guesses on 10 true-false questions. Find the probability that she gets exactly 2 wrong.

[**a**] 45/1024 (b) 4/5 (c) 1/8 (d) 33/1024 (e) 45/512

Solution to 28. There are 2^{10} possible answer sets. The 10 questions can be marked with 8 correct and 2 incorrect in $1+2+\cdots+9 = 9 \cdot 10/2$ ways. The probability is $45/1024$.

29. **(Models)** Let $f(a)$ be the circumference in feet for a circle of area a square feet. Which answer below is the circumference of a circle of diameter 4?

(a) $f(2\pi)$ (b) $f(4\pi)/\pi$ (c) $f(\pi)$ **[d]** $\pi f(4/\pi)$ (e) $4f(\pi/2)$

Solution to 29. Use $a = \pi r^2$ and $c = 2\pi r$. By the area formula, $f(a) = c = 2\pi r = 2\sqrt{\pi}\sqrt{\pi r^2} = 2\sqrt{a\pi}$. The area of a circle of diameter 4 (radius 2) is 4π. The answers given are (a) 2π, (b) $2\sqrt{2}$, (c) 4, (d) 4π and (e) $4\sqrt{2}\pi$. The match is (d).

30. **(Logarithms)** Assume $y > 0$ and $x > 0$. Solve for y: $\log_4 x^2 + 5\log_4 y^2 = 4\log_{16} y^6$

(a) $5x^3$ (b) 2^x (c) 4^x (d) x^2 **[e]** x

Solution to 30. Use $\log_b u = \log_c u/\log_c b$ to write the equation as $\log_4(x^2y^{10}) = 4\log_4 y^6/\log_4 16$. Then write as $\log_4(x^2y^{10}) = \log_4 y^{12}$ to get $x^2y^{10} = y^{12}$. Cancel to obtain $y = x$.

See *infra* for problems 31-40.

1996 Senior Exam Grade 11 with Key and Solutions

Answers to 1996 Exam Grade 11			
1. **(d)**	11. **(d)**	21. **(c)**	31. **(c)**
2. **(b)**	12. **(d)**	22. **(d)**	32. **(a)**
3. **(d)**	13. **(e)**	23. **(e)**	33. **(a)**
4. **(d)**	14. **(c)**	24. **(d)**	34. **(e)**
5. **(a)**	15. **(b)**	25. **(d)**	35. **(a)**
6. **(e)**	16. **(e)**	26. **(c)**	36. **(a)**
7. **(a)**	17. **(c)**	27. **(c)**	37. **(b)**
8. **(a)**	18. **(e)**	28. **(a)**	38. **(d)**
9. **(e)**	19. **(a)**	29. **(c)**	39. **(c)**
10. **(a)**	20. **(b)**	30. **(b)**	40. **(d)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Coolant)** Mix 2, 5 and 7 liters of coolant at temperatures 50°, 67° and 73° , respectively, to create a mixture at x degrees. Find x to the nearest degree.

(a) 946 (b) 66 (c) 67 **[d]** 68 (e) 70

Solution to 1. The mixture equation is $(2 + 5 + 7)x = 2(50) + 5(67) + 7(73)$. The answer is $x = 946/14 = 67.57$. The closest answer is 68.

2. **(Tables)** Find an entry $x > 0$ that completes the table, given $y = \dfrac{x}{x+4}$.

x	0	1	
$3xy$	0	$\frac{3}{5}$	$\frac{1}{13}$

(a) 1/5 **[b]** 1/3 (c) 2/5 (d) 3/5 (e) 2/3

Solution to 2. Solve $3x^2/(x+4) = 1/13$ for x and compare to the table to select 1/3. Alternatively, try each answer in the formula $3xy$.

3. **(Perfect Cubes)** What fraction of the positive odd integers less than 2199 are perfect cubes?

(a) 1/100 (b) 11/2200 (c) 13/1100 **[d]** 7/1100 (e) 7/2200

Solution to 3. The odd perfect cubes are 1^3, 3^3, ..., 13^3. The odd numbers are 1, 3, 5, ..., 2199. The counts are 7 and 1100 for fraction 7/1100.

4. **(Statistics)** Find the average of the mode and the mean for the data set 49, 54, 61, 49, 54, 49, 64 and 60.

(a) 49 (b) 50 (c) 51 **[d]** 52 (e) 55

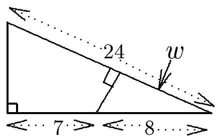
Solution to 4. The mode is the most frequently listed value: $\overline{m} = 49$. The mean is the average $\overline{x} = 55$. Then $\frac{1}{2}(\overline{m} + \overline{x}) = \frac{1}{2}(49 + 55) = 52$

5. **(Trigonometry)** A triangle has sides of length 4, 5 and 8.5 units. Find the cosine of the largest angle.

[a] $-\frac{25}{32}$ (b) $-\frac{3}{4}$ (c) $-\frac{13}{16}$ (d) $-\frac{27}{32}$ (e) None

Solution to 5. The largest angle is opposite the longest side. Apply the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$ to obtain $\cos C = (a^2 + b^2 - c^2)/(2ab)$. Then $\cos C = -25/32$.

6. **(Triangle)** Find side w in the triangle.



(a) 6 (b) 16/3 (c) 11/2 (d) 14/3 **[e]** 5

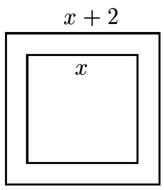
Solution to 6. The small right triangle is similar to the large right triangle. Therefore, $w/8 = (7 + 8)/24$ and $w = 5$.

7. **(Triangle)** Classify the triangle ABC with vertices $A(2, -4)$, $B(-3, 1)$ and $C(1, 6)$.

[a] obtuse scalene (b) acute scalene
(c) right scalene (d) isosceles (e) None

Solution to 7. By the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ the sides have lengths $\sqrt{50}$, $\sqrt{41}$ and $\sqrt{101}$, so it's scalene. The angle at B is not a right angle because the Pythagorean relation fails: $\sqrt{50}^2 + \sqrt{41}^2 \neq \sqrt{101}^2$. In fact, this relation says that the angle at B is obtuse.

8. **(Photo)** A square picture frame is to be designed as in the figure so that $\frac{2}{3}$ of the total area is used by the inner frame. Find the area of the inner frame in square units.



[a] $40 + 16\sqrt{6}$ (b) $40 + 23\sqrt{3}$ (c) $40 + 28\sqrt{2}$ (d) $33 + 19\sqrt{6}$ (e) None

Solution to 8. The inner square has area x^2 and the outer one has area $(x + 2)^2$, therefore $\frac{2}{3}(x + 2)^2 = x^2$. Solving for $x = 4 + 2\sqrt{6}$ results in inner area $x^2 = 40 + 16\sqrt{6}$.

9. **(Triangle)** Sides a and b of a triangle satisfy the equation $6a^2 + 7ab + 6b^2 = 30$. Side c is divided by the bisector of the opposite angle in the ratio 3:2. Find the smaller of a and b.

(a) $\sqrt{7/6}$ (b) $\sqrt{7/5}$ (c) $3/2$ (d) $2/3$ [**e**] 1

Solution to 9. The relation $a/b = 3/2$ is given by the bisector hypothesis. Solve for $a = 3b/2$ and insert into the quadratic equation to get $30b^2 = 30$. Then $b = 1$ and $a = 3/2$.

10. **(Rectangle)** Given a rectangle of area 1248 square units and diagonal 52 units, determine the ratio of its sides.

[**a**] $\frac{3}{2}$ (b) $\frac{6}{5}$ (c) $\frac{4}{3}$ (d) $\frac{7}{5}$ (e) $\frac{9}{7}$

Solution to 10. Let x and y be the two sides. Then $x^2 + y^2 = 52^2$ and $xy = 1248$. Hence $(x+y)^2 = x^2 + 2xy + y^2 = 5200$ and $x + y = 20\sqrt{13}$. Then $1248 = xy = x(20\sqrt{13} - x)$ is a quadratic equation for x with solutions determined by the FOIL factorization $(x - 12\sqrt{13})(x - 8\sqrt{13}) = 0$. The ratio is $12/8 = 3/2$.

11. **(Navigation)** A lighthouse and a ship are x *feet* apart, both at sea level, with the light located 400 feet above the sea. The *angle of depression* is 15°, from the light to the ship. Approximate x.

(a) 1100 (b) 1451 (c) 1472 [**d**] 1493 (e) None

Solution to 11. Using right triangle geometry, $400/x = \tan(15)$, giving $x = 400\cot(15) = 1493$, approximately. The value of $\sin(\theta/2)$ can be obtained from the half-angle identity. Similarly for $\cos(\theta/2)$. Therefore, the value of $\tan(15)$ can be deduced from $\sin(30)$ and $\cos(30)$.

12. **(Skid)** Jane measured the skid marks when the brakes of her Justy were applied at different speeds. Use a simple data fitting model to estimate how far in feet the Justy would skid if the brakes were applied at 70 mph.

ft	mph
44	30
78	40
123	50
176	60

(a) 210 (b) 220 (c) 230 [**d**] 240 (e) 250

Solution to 12. The graph of the data looks quadratic. The best model seems to be $y = kx^2$ where y is the skid length in feet and x is the speed in mph. The first data point gives $k = 44/30^2 = 0.049$. Testing $y = 0.049x^2$ reproduces the table. The prediction is y at $x = 70$, which is $y = 0.049(70)^2 = 240.1$ feet. Other values of k can be obtained from the other data items, but there is not much change and the logical answer is 240.

13. **(Intercept)** Find the y-intercept of the line through $(5, 2)$ and perpendicular to $5x - 7y = 17$.

(a) -7 (b) -8 (c) 7 (d) 8 [**e**] 9

Solution to 13. The line is $y - (2) = m(x - 5)$ where $-1/m$ is the slope of $5x - 7y = 17$. Then $-1/m = 5/7$ giving $m = -7/5$. Set $x = 0$ in the equation to get the y-intercept $y = 2 + m(0 - 5) = 9$.

14. **(Water Tank)** A tank has a drain rate of 10% of capacity per hour. Its fill rate is 15% of capacity per hour. When 2/3 full and draining, about how many hours does it take to fill?

(a) $6\frac{1}{3}$ (b) $6\frac{4}{9}$ [**c**] $6\frac{2}{3}$ (d) $5\frac{2}{3}$ (e) $5\frac{1}{3}$

Solution to 14. The fill time t satisfies $t(0.15x - 0.1x) + 2x/3 = x$ where x is the tank capacity. Cancel x and solve for $t = 20/3$.

15. **(Complex)** Simplify the number $8(1+i)(1-i)^{-6}$.

(a) $1 + i$ [**b**] $1 - i$ (c) $2 + 2i)$ (d) $8i$ (e) i

Solution to 15. First, $(1-i)^2 = 1 - 2i + i^2 = -2i$, so $(1-i)^6 = (-2i)^3 = -8i^3 = 8i$. The answer is $8(1+i)/(8i) = (i + i^2)/i^2 = 1 - i$.

16. **(Monotone)** On which interval below is $y = x^2 - 5x + 6$ strictly increasing?

(a) $1 < x < 5$ (b) $0 < x < 5$ (c) $2 < x < 3$
(d) $2 < x < 4$ [**e**] $3 < x < 5$

Solution to 16. Factor into the form $y = (x - \frac{5}{2})^2 + 6 - \frac{25}{4}$. Then y increases to the right of the vertex point $x = 2.5$.

17. **(Linear System)** Compute the value of $80x$:

$$\left\{\begin{array}{rcl} x - 2y - 3z & = & 2 \\ x - 4y + 3z & = & a \\ -3x + 5y + 4z & = & 0 \end{array}\right.$$

(a) $5a - 70$ (b) $-130 - 25a$ [**c**] $-310 - 35a$
(d) $1 + a$ (e) None

Solution to 17. Cramer's Rule applies. The elimination method applies. The solution is $80x = -310 - 35a$, $80y = -130 - 25a$, $80z = 5a - 70$.

18. **(Distance)** Find the distance between the points $(-1, 4, 2)$ and $(1, 2, -2)$.

(a) $2\sqrt{2}$ (b) $2\sqrt{3}$ (c) 4 (d) $2\sqrt{5}$ [**e**] $2\sqrt{6}$

Solution to 18. Use the distance formula $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ to obtain $\sqrt{(1 - (-1))^2 + (2 - 4)^2 + (-2 - 2)^2} = 2\sqrt{6}$.

19. **(Sequence)** Find the value of a_{12}/b_9, given $a_n = 3200(0.8)^{2n+1}$ and $b_n = 1600(0.2)^{3n-2}$.

[**a**] 2^{51} (b) 4^{25} (c) 4^{26} (d) 2^{26} (e) None

Solution to 19. The values are $a_{12} = 3200(0.8)^{25}$ and $b_9 = 1600(0.2)^{25}$. Then $a_{12}/b_9 = 2(0.8)^{25}/(0.2)^{25} = 2(0.8/0.2)^{25} = 2(4^{25}) = 2^{51}$.

20. **(Lines)** Find the distance from the point $(3, 5)$ to the line $12y = -5x - 3$.

(a) $5\frac{12}{13}$ [**b**] 6 (c) $6\frac{1}{13}$ (d) $6\frac{2}{3}$ (e) $6\frac{2}{13}$

Solution to 20. The standard form of the line is $5x + 12y + 3 = 0$. Using a classical formula $\sqrt{A^2 + B^2}d = Ax_1 + By_1 + C$ for the distance d gives the answer $\sqrt{5^2 + 12^2}d = 5(3) + 12(5) + 3 = 78$ or $13d = 78$, and finally $d = 6$.

21. (**Asymptote**) Find the slant or oblique asymptote for $y = \dfrac{2x^3 + x^2 + x + 1}{x^2 + 1}$.

(a) $y = x + 1$ (b) $y = x - 1$ [**c**] $y = 2x + 1$
(d) $y = 2x - 1$ (e) $y = x - 2$

Solution to 21. Divide $2x^3 + x^2 + x + 1$ by $x^2 + 1$ using long division. The quotient is $2x + 1$ and the remainder is $-x/(x^2 + 1)$. The slant asymptote is $y = 2x + 1$.

22. (**Roots**) Apply the *Rational Root Theorem* to select from the list below a *candidate* for a rational root of $7x^9 - 5x^7 + 2x^3 + 210 = 0$.

(a) 9 (b) $\frac{8}{7}$ (c) $\frac{5}{14}$ [**d**] $-\frac{6}{7}$ (e) 11

Solution to 22. A possible rational root must be a factor of $210 = (2)(3)(5)(7)$ divided by a factor of 7. The only one that qualifies from the given list is -6. A candidate is not necessarily a root!

23. (**Disk**) Select from the list below a point which is inside the circle of center $(-1, 2)$ and radius 3 and outside the circle of center $(-1, 1)$ and radius 2.

(a) $(-2, 1)$ (b) $(-1 - \sqrt{2}, 1)$ (c) $(-1 - \sqrt{3}, 1)$
(d) $(-3, 1)$ [**e**] $(-1 - \sqrt{5}, 1)$

Solution to 23. The inequalities are $(x + 1)^2 + (y - 2)^2 < 9$ and $(x + 1)^2 + (y - 1)^2 > 4$. Try the points, one at a time.

24. (**Symmetry**) Find the axis of symmetry for the parabola that passes through the points $(-3, 47)$, $(2, 7)$ and $(3, 23)$.

(a) $x = -\frac{1}{4}$ (b) $x = -\frac{1}{2}$ (c) $x = \frac{1}{4}$ [**d**] $x = \frac{1}{2}$
(e) $x = \frac{3}{4}$

Solution to 24. The axis of symmetry is the vertical line through the vertex. The equation of the parabola is $y = a + b(x + 3)) + c(x + 3)(x - 2)$ (any quadratic will do). Stuff the data points into this equation. Then $a = 47$, $b = -8$ and $c = 4$. The equation is $y = 47 - 8(x + 3) + 4(x + 3)(x - 2)$. The vertex is at $x = 1/2$.

25. (**Partial fractions**) Find $A + B + C$:
$$\frac{x - 9}{(x^2 - 4x + 4)(x + 3)} = \frac{A}{(x - 2)^2} + \frac{B}{x - 2} + \frac{C}{x + 3}.$$

(a) $\frac{7}{5}$ (b) $\frac{12}{25}$ (c) $-\frac{12}{25}$ [**d**] $-\frac{7}{5}$ (e) 0

Solution to 25. Multiply by $(x - 2)^2$ and cancel like terms to get $(x - 9)/(x + 3) = A + B(x - 2) + C(x - 2)^2/(x + 3)$. Let $x = 2$. Then $-7/5 = A + 0 + 0$ giving $A = -7/5$. Similarly, $C = -12/25$. Substitute $x = 0$ to get an equation for B, obtaining $B = 12/25$. Then $A + B + C = -7/5$.

26. (**Intersect**) The graphs $y = 4x^3 + 9x^2 + 2x + 5$ and $y = 3x + 8$ intersect at three x-values $x = x_1$, $x = x_2$ and $x = x_3$. Find $x_1 + x_2 + x_3$ to the nearest hundredth.

(a) -2.27 (b) -2.26 [**c**] -2.25 (d) -2.24
(e) -2.23

Solution to 26. The fastest method to solve this problem is to apply the theorem on the sum of the roots of a polynomial. The polynomial is $4x^3 + 9x^2 + 2x + 5 - (3x + 8) = 4x^3 + 9x^2 - x - 3$. The sum of the roots is $\frac{1}{4}(-1)(9) = -\frac{9}{4} = -2.25$.

27. (**Inverse**) Let $f(x) = (x + 1)/(x + 2)$. Compute the value of the inverse of f at $y = \sqrt{2}/(1 + \sqrt{2})$.

(a) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ (b) $\sqrt{2} + 1$ [**c**] $\sqrt{2} - 1$ (d) $\sqrt{2}$
(e) $-\sqrt{2}$

Solution to 27. Let $y = (x + 1)/(x + 2)$. Solve for x in terms of y. Then $x = (1 - 2y)/(y - 1)$. Let $y = \sqrt{2}/(1 + \sqrt{2})$. Then $x = \sqrt{2} - 1$.

28. (**Maximum**) Find the maximum area for a rectangle of sides x and y subject to the restraint $y + 2x = 5$.

[**a**] $\frac{25}{8}$ (b) $\frac{25}{4}$ (c) $\frac{75}{8}$ (d) $\frac{25}{2}$ (e) $\frac{5}{4}$

Solution to 28. Solve for $y = 5 - 2x$, insert into the area formula xy to get the function $f(x) = x(5 - 2x)$, to be maximized. The maximum is at the vertex of the parabola, which is at $x = 5/4$. Then $f(x) = x(5 - 2x) = (5/4)(5 - 5/2) = 25/8$.

29. (**Radius**) Find the radius of the inscribed circle to the nearest tenth for a triangle of sides 12, 15 and 25.

(a) 2.0 (b) 2.2 [**c**] 2.4 (d) 2.6 (e) 2.8

Solution to 29. This problem is solved using Heron's formula $r = \frac{1}{s}\sqrt{s(s - a)s - b)(s - c)}$ where $s = (a + b + c)/2$ is the half–perimeter and a, b, c are the sides of the triangle. In this problem, $r = \frac{1}{26}\sqrt{(26)(14)(11)(1)} = 2.43373723$.

30. (**Planes**) Find the distance to the nearest tenth from $(24, -5, 2)$ to the plane $4x - 12y - 3z + 6 = 0$.

(a) 14.4 [**b**] 12 (c) 10.8 (d) 9 (e) None

Solution to 30. Apply the standard formula $\sqrt{A^2 + B^2 + C^2}d = Ax_1 + By_1 + Cz_1 + D$ for the distance d from a point to a plane. The result is $\sqrt{4^2 + 12^2 + 3^2}\, d = 4(24) - 12(-5) - 3(2) + 6$, which reduces to $d = 156/13 = 12$.

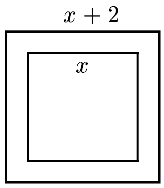
See *infra* for problems 31-40.

1996 Senior Exam Grade 12 with Key and Solutions

Answers to 1996 Exam Grade 12			
1. **(a)**	11. **(b)**	21. **(c)**	31. **(c)**
2. **(a)**	12. **(a)**	22. **(b)**	32. **(a)**
3. **(c)**	13. **(c)**	23. **(a)**	33. **(a)**
4. **(d)**	14. **(c)**	24. **(c)**	34. **(e)**
5. **(d)**	15. **(d)**	25. **(b)**	35. **(a)**
6. **(b)**	16. **(d)**	26. **(d)**	36. **(a)**
7. **(a)**	17. **(d)**	27. **(c)**	37. **(b)**
8. **(e)**	18. **(c)**	28. **(e)**	38. **(d)**
9. **(a)**	19. **(c)**	29. **(d)**	39. **(c)**
10. **(d)**	20. **(a)**	30. **(a)**	40. **(d)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Photo)** A square picture frame is to be designed as in the figure so that $\frac{2}{3}$ of the total area is used by the inner frame. Find the area of the inner frame in square units.



[**a**] $40+16\sqrt{6}$ (b) $40+23\sqrt{3}$ (c) $40+28\sqrt{2}$ (d) $33+19\sqrt{6}$ (e) None

Solution to 1. The inner square has area x^2 and the outer one has area $(x+2)^2$, therefore $\frac{2}{3}(x+2)^2 = x^2$. Solving for $x = 4+2\sqrt{6}$ results in inner area $x^2 = 40+16\sqrt{6}$.

2. **(Series)** Find the sum the infinite series

$$0.002 + 0.0002 + 0.00002 + \cdots$$

[**a**] $\frac{2}{900}$ (b) $\frac{1}{300}$ (c) $\frac{5}{900}$ (d) $\frac{4}{900}$ (e) None

Solution to 2. The series is geometric with common ratio r. Write it as $0.002(1+r+r^2+\cdots)$ where $r = 0.1$. Then the sum is $0.002/(1-r) = 0.002(10/9) = 2/900$.

3. **(Max-Min)** Find the maximum value of $y = 16x^2 - x^3$ on $0 \le x < \infty$ to the nearest integer.

(a) 598 (b) 605 [**c**] 607 (d) 609 (e) 611

Solution to 3. The function y takes on positive values and is negative for large values of x. Therefore, its maximum is assumed on some closed interval $[0, R]$. If R is large, then y is negative at $x = R$ and y is zero at $x = 0$, so the maximum must be in $(0, R)$. A maximum occurs at a critical point of y. The derivative is $y' = 32x - 3x^2 = x(32-3x)$. The interior critical point is $x = 32/3$. The maximum value at $x = 32/3$ is $y = (32/3)^2(16 - 32/3) = \frac{2^{14}}{3^3} = 606.81$. A graphing calculator, properly used, can zero on the value without using calculus. This kind of problem was nearly the most successful 12th grade problem in 1995!

4. **(Lines)** Along the line through $(3,4)$ and parallel to $5x+4y = 5$, find the value of y when $x = -3$.

(a) 9 (b) 10 (c) 11 [**d**] 23/2 (e) None

Solution to 4. A parallel line is $5x+4y = c$ for any c. To insure $(3,4)$ on this line, take $c = 5(3)+4(4) = 31$. Then at $x = -3$, $4y = c - 5x = 31 - 5(-3) = 46$ and finally $y = 23/2$.

5. **(Conics)** Find the center of the circle whose equation is $x^2+y^2+2x+8y = 8$.

(a) $(1,-4)$ (b) $(1,4)$ (c) $(-1,4)$ [**d**] $(-1,-4)$ (e) $(2,-4)$

Solution to 5. Complete the square and write as $(x+1)^2+(y+4)^2 = 8+1+16$. Then the center is $(-1,-4)$. Alternatively, using calculus, the center is where the partial derivatives in x and y vanish.

6. **(Area)** Find the area bounded by $x = 0$, $x = 1$, $y = 0$ and $y = x^2(1+x^2)$.

(a) $\frac{7}{15}$ [**b**] $\frac{8}{15}$ (c) $\frac{3}{5}$ (d) $\frac{2}{3}$ (e) $\frac{11}{15}$

Solution to 6. The function $x^2(1+x^2)$ is positive on $[0,1]$. So the area is $\int_0^1 x^2(1+x^2)dx = [x^3/3 + x^5/5]|_0^1 = \frac{1}{3}+\frac{1}{5} = 8/15$.

7. **(Sequence)** Find the value of a_{12}/b_9, given $a_n = 3200(0.8)^{2n+1}$ and $b_n = 1600(0.2)^{3n-2}$.

[**a**] 2^{51} (b) 4^{25} (c) 4^{26} (d) 2^{26} (e) None

Solution to 7. The values are $a_{12} = 3200(0.8)^{25}$ and $b_9 = 1600(0.2)^{25}$. Then $a_{12}/b_9 = 2(0.8)^{25}/(0.2)^{25} = 2(0.8/0.2)^{25} = 2(4^{25}) = 2^{51}$.

8. **(Minima)** Determine a value of x for which the cubic $y - 1 = 5x^3 - x^2$ has a local minimum.

(a) $\frac{1}{15}$ (b) $\frac{4}{15}$ (c) $\frac{1}{3}$ (d) 0 [**e**] None

Solution to 8. Calculus method: the critical points are roots of the quadratic $15x^2 - 2x$ ($x = 0$ and $x = 2/15$ are critical points).

A graphical method, without calculus: the graph suggests a relative minimum near $2/15$.

9. **(Sequences)** The first four terms of an arithmetic sequence are 1, a, b, 16. Find term 47.

[**a**] 231 (b) 235 (c) 239 (d) 243 (e) 247

Solution to 9. The common difference x satisfies $x = a-1 = b-a = 16-b$. Adding these gives $3x = (a-1)+(b-a)+(16-b) = 15$ or $x = 5$. The terms are 1, 6, 11, 16, ... The n^{th} term is $x_n = 1+5(n-1)$ so $x_{47} = 1+5(46) = 231$.

10. **(Tangent)** Find the slope of the line tangent at $x = e^{-3}$ to the curve $y = x\ln(x^2)$.

(a) -1 (b) -2 (c) -3 [**d**] -4 (e) None

Solution to 10. The derivative is $y' = 2(1+\ln(x)) = 2+2\ln(e^{-3}) = 2+2(-3)\ln(e) = -4$.

11. **(Normal)** A function $f(x)$ has slope $-1/5$ at $(1,2)$. Find the y-intercept of the normal line at $(1,2)$.

(a) 3 [**b**] -3 (c) 2 (d) -2 (e) 1

Solution to 11. The tangent line is given by $y = 2-(x-1)/5$. The normal line is given by $y = 2+5(x-1)$, because the slope of the normal line is the negative reciprocal of the slope of the tangent line. The y-intercept is the solution of the equation $y = 2+5(0-1)$. Then $y = -3$.

12. **(Quadratics)** Given real a and b, which inequality implies $x^2+bx+4a = 0$ has a complex root?

[**a**] $b^2 < 16a$ (b) $b^2 < 4a$ (c) $a > 4b^2$ (d) $b^2 > 4a$ (e) $a^2 < 4b$

Solution to 12. Let $A = 1$, $B = b$, $C = 4a$ in $Ax^2+Bx+C = 0$. The roots are complex if and only if the discriminant $B^2 - 4AC = b^2 - 4(1)(2a)$ is negative. Then $b^2 < 16a$.

13. **(Ellipse)** Find an axis of symmetry of the ellipse $4x^2 + 25y^2 + 250y + 525 = 0$.

(a) $y = 5$ (b) $x = 5$ [**c**] $y = -5$ (d) $x = -5$ (e) $y = 0$

Solution to 13. Complete the square to write the equation in the equivalent form $4x^2 + 25(y^2 + 10y + 25) - 25^2 + 525$ and finally as $4x^2 + 25(y+5)^2 = 100$. The center is $(0, -5)$. Axes of symmetry are $x = 0$ and $y = -5$.

14. **(Arcsine)** Simplify $\cos(\mathrm{Sin}^{-1}(x/2))$ for $|x| \le 2$.

(a) $x/2$ (b) $1 - x^2/4$ [**c**] $\sqrt{1 - x^2/4}$ (d) $4/(4 + x^2)$ (e) None

Solution to 14. The principal arcsine is defined on $[-1, 1]$ and takes values in $[-\pi/2, \pi/2]$. Let $\theta = \mathrm{Sin}^{-1}(x/2)$. Then $\sin\theta = x/2$ and θ between $-\pi/2$ and $\pi/2$ implies the cosine is positive and $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - (x/2)^2}$.

15. **(Derivative)** Let f be defined on $(-\infty, \infty)$ and differentiable with $f'(3) = \pi$ and $f(3) = 1$. Find $\frac{d}{dx}[f(x)f(x^2 - 6)]$ at $x = 3$.

(a) 4π (b) 5π (c) 6π [**d**] 7π (e) None

Solution to 15. Let $z = f(x)f(x^2 - 6)$. By the product rule it follows that $z' = f'(x)f(x^2 - 6) + f(x)[f(x^2 - 6)]'$. By the chain rule, $[f(x^2 - 6)]' = f'(x^2 - 6)(2x)$. At $x = 3$, the value is $z' = f'(3)f(3) + f(3)f'(3)(2)(3) = \pi + 6\pi = 7\pi$.

16. **(Inequalities)** Find the maximum of $4x + 5y + 3$ subject to the inequalities $x \ge 0$, $y \ge 0$, $x + y \le 3$ and $3x + y \le 6$.

(a) 11 (b) 12 (c) 16 [**d**] 18 (e) 19

Solution to 16. The function $z = 4x + 5y + 3$ is a plane in R^3. The inequalities describe a polygon in the plane. The maximum occurs at a vertex of the polygon: $(0,0)$, $(2,0)$, $(3/2, 3/2)$, $(0,3)$. The values of z are 3, 11, 16.5, 18, respectively. The maximum of z is 18.

17. **(Partial fractions)** Find $A + B + C$:

$$\frac{x - 9}{(x^2 - 4x + 4)(x + 3)} = \frac{A}{(x - 2)^2} + \frac{B}{x - 2} + \frac{C}{x + 3}.$$

(a) $\frac{7}{5}$ (b) $\frac{12}{25}$ (c) $-\frac{12}{25}$ [**d**] $-\frac{7}{5}$ (e) 0

Solution to 17. Multiply by $(x - 2)^2$ and cancel like terms to get $(x - 9)/(x + 3) = A + B(x - 2) + C(x-2)^2/(x+3)$. Let $x = 2$. Then $-7/5 = A + 0 + 0$ giving $A = -7/5$. Similarly, $C = -12/25$. Substitute $x = 0$ to get an equation for B, obtaining $B = 12/25$. Then $A + B + C = -7/5$.

18. **(Intersect)** The graphs $y = 4x^3 + 9x^2 + 2x + 5$ and $y = 3x + 8$ intersect at three x-values $x = x_1$, $x = x_2$ and $x = x_3$. Find $x_1 + x_2 + x_3$ to the nearest hundredth.

(a) -2.27 (b) -2.26 [**c**] -2.25 (d) -2.24 (e) -2.23

Solution to 18. The fastest method to solve this problem is to apply the theorem on the sum of the roots of a polynomial. The polynomial is $4x^3 + 9x^2 + 2x + 5 - (3x + 8) = 4x^3 + 9x^2 - x - 3$. The sum of the roots is $\frac{1}{4}(-1)(9) = -\frac{9}{4} = -2.25$.

19. **(Inverse)** Let $f(x) = (x + 1)/(x + 2)$. Compute the value of the inverse of f at $y = \sqrt{2}/(1 + \sqrt{2})$.

(a) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ (b) $\sqrt{2} + 1$ [**c**] $\sqrt{2} - 1$ (d) $\sqrt{2}$ (e) $-\sqrt{2}$

Solution to 19. Let $y = (x + 1)/(x + 2)$. Solve for x in terms of y. Then $x = (1 - 2y)/(y - 1)$. Let $y = \sqrt{2}/(1 + \sqrt{2})$. Then $x = \sqrt{2} - 1$.

20. **(Maximum)** Find the maximum area for a rectangle of sides x and y subject to the restraint $y + 2x = 5$.

[**a**] $\frac{25}{8}$ (b) $\frac{25}{4}$ (c) $\frac{75}{8}$ (d) $\frac{25}{2}$ (e) $\frac{5}{4}$

Solution to 20. Solve for $y = 5 - 2x$, insert into the area formula xy to get the function $f(x) = x(5 - 2x)$, to be maximized. The maximum is at the vertex of the parabola, which is at $x = 5/4$. Then $f(x) = x(5 - 2x) = (5/4)(5 - 5/2) = 25/8$.

21. **(Radius)** Find the radius to the nearest tenth of the inscribed circle for a triangle of sides 12, 15 and 25.

(a) 2.0 (b) 2.2 [**c**] 2.4 (d) 2.6 (e) 2.8

Solution to 21. This problem is solved using Heron's formula $r = \frac{1}{s}\sqrt{s(s - a)s - b)(s - c)}$ where $s = (a + b + c)/2$ is the half–perimeter and a, b, c are the sides of the triangle. In this problem, $r = \frac{1}{26}\sqrt{(26)(14)(11)(1)} = 2.43373723$.

22. **(Planes)** Find the distance from $(24, -5, 2)$ to the plane $4x - 12y - 3z + 6 = 0$, to the nearest tenth.

(a) 14.4 [**b**] 12 (c) 10.8 (d) 9 (e) None

Solution to 22. Apply the standard formula $\sqrt{A^2 + B^2 + C^2}d = Ax_1 + By_1 + Cz_1 + D$ for the distance d from a point to a plane. The result is $\sqrt{4^2 + 12^2 + 3^2}\,d = 4(24) - 12(-5) - 3(2) + 6$, which reduces to $d = 156/13 = 12$.

23. **(Asymptotes)** The curve $2xy = x^2 - 6x + 4y + 10$ has slant (oblique) and vertical asymptotes. One is $y = x/2 - 2$. Find the other.

[**a**] $x = 2$ (b) $y = 2$ (c) $y = x/2 + 1$ (d) $y = 2x$ (e) None

Solution to 23. In normal form, $y = (x^2 - 6x + 10)/(2x - 4) = x/2 - 2 + 2/(2x - 4)$. The vertical asymptote is $x = 2$ and the slant asymptote is $y = x/2 - 2$.

24. **(Cofactor)** Find the cofactor (signed minor) of row 3 and column 2 in the determinant $\begin{vmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{vmatrix}$.

(a) ab (b) $-b^2$ [**c**] b^2 (d) a^2 (e) $a^3 + b^3$

Solution to 24. The minor is $\begin{vmatrix} a & b \\ b & 0 \end{vmatrix} = -b^2$. The sign is $(-1)^{3+2} = -1$. The cofactor is $(-1)(-b^2)$.

25. **(Integral)** Let f be the unique quadratic polynomial which passes through $(0,1)$, $(1,-1)$ and $(2,3)$. Find $\int_0^2 |f(x)|^2\,dx$.

(a) $\frac{31}{15}$ **[b]** $\frac{34}{15}$ (c) $\frac{7}{3}$ (d) $\frac{8}{3}$ (e) None

Solution to 25. The polynomial is $y = 3x^2 - 5x + 1$. To find this polynomial, let $f(x) = a+bx+cx(x-1)$ and set up three equations in three unknowns a, b, c using the given data points. Solve for a, b, c by elimination or Cramer's rule. The integral is $\int_0^2 (3x^2-5x+1)^2 dx = (9/5)x^5-(15/2)x^4+(31/3)x^3-5x^2+x|_0^2 = 34/15$.

26. **(Series)** Sum the series $\sum_{n=1}^{\infty} 5^{-2n+2}\,4^{2n-3}$.

(a) $\frac{25}{16}$ (b) $\frac{25}{18}$ (c) $\frac{25}{24}$ **[d]** $\frac{25}{36}$ (e) None

Solution to 26. Write the term as $(5^2)(4^{-3})5^{-2n}\,4^{2n} = c(16/25)^n$ where $c = 5^2/4^3$. Then the summation, by geometric series formulas, is $c(16/25)/(1-(16/25) = 16c/9 = 25/36$.

27. **(Inverse)** Find the determinant of $2A^{-2} + A^{-3}$, given

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & -x & -1 \end{pmatrix}$$

(a) 3 (b) x **[c]** $\frac{4x+7}{27}$ (d) $\frac{4x+7}{9}$ (e) $\frac{4x+7}{3}$

Solution to 27. The determinant rule $\det(AB) = \det(A)\det(B)$ can be applied to $2A^{-2} + A^{-3} = A^{-3}(I+2A)$ to give the answer as the product of $\det(A^{-3})$ and $\det(I+2A)$. The first factor is $1/(\det(A))^3 = 1/27$ because of $\det(A) = 3$ and determinant rules. The second factor is a direct evaluation,

$$I + 2A = \begin{pmatrix} 3 & 4 & 2 \\ 2 & 3 & 2 \\ 4 & -2x & -1 \end{pmatrix}$$

which has value $4x+7$. The product of the factors is $(4x+7)/27$.

28. **(Triangle)** Sides a and b of a triangle satisfy the equation $6a^2 + 7ab + 6b^2 = 30$. Side c is divided by the bisector of the opposite angle in the ratio 3:2. Find the smaller of a and b.

(a) $\sqrt{7/6}$ (b) $\sqrt{7/5}$ (c) $3/2$ (d) $2/3$ **[e]** 1

Solution to 28. The relation $a/b = 3/2$ is given by the bisector hypothesis. Solve for $a = 3b/2$ and insert into the quadratic equation to get $30b^2 = 30$. Then $b = 1$ and $a = 3/2$.

29. **(Probability)** A coin is tossed three times. Find the probability that at most two heads show given that at least one head shows.

(a) $3/8$ (b) $5/8$ (c) $5/7$ **[d]** $6/7$ (e) $3/4$

Solution to 29. This can be solved using conditional probability $P(A/B) = P(A \text{ and } B)/P(B)$. A simpler method is to use the reduced sample space of events *htt*, *tht*, *tth*, *hht*, *hth*, *thh*, *hhh* consisting of all events with at least one head. The probability is then the number of events with one or two h's divided by the total number of events: $6/7$.

30. **(Skeet)** Jack normally hits 4 skeets out of 6 attempts. Find the probability that he hits exactly 2 in 4 attempts.

[a] $24/81$ (b) $(2/3)^4$ (c) $32/81$ (d) $\frac{6!}{4!2!}\frac{4}{81}$
(e) None

Solution to 30. By advanced textbook methods the probability is Ca^kb^{n-k} where $n=4$, $k=2$, $C = \begin{pmatrix} n \\ k \end{pmatrix} = 6$, $a = 4/6$ and $b = 2/6$. See Merrill Algebra 2 (1992), p 720. The answer is $(6)(4/6)^2(2/6)^2 = 24/81$. An elementary solution starts with the probability of a successful single trial of two hits and two misses. The probability of success is $2/3$ and of failure $1/3$ (4 hits out 6). Two hits and two misses has probability $\frac{2}{3}\frac{2}{3}\frac{1}{3}\frac{1}{3} = 4/81$. There are 6 ways for two hits and two misses: $HHMM$, $HMHM$, $HMMH$, $MHHM$, $MHMH$, $MMHH$. Adding the probabilities of each single event gives $24/81$.

Problems 31–40 have a higher weight in the Senior competition between grades 10, 11 and 12.

31. **(Polynomials)** Let p be a cubic polynomial in x with roots -2, 2 and 5. Assume $p = 20$ when $x = 0$. Find p when $x = 1$.

(a) 10 (b) 11 **[c]** 12 (d) 13 (e) 14

Solution to 31. By the factor theorem, $(x+2)$, $(x-2)$ and $(x-5)$ are factors of p. The polynomial p must be a constant multiple of $(x+2)(x-2)(x-5)$. Briefly, this means $p = c(x+2)(x-2)(x-5)$ for some constant c. In expanded form, $p = c(x^3 - 5x^2 - 4x + 20)$. Since $p = 20$ at $x = 0$, then $c = 1$ and $p = x^3 - 5x^2 - 4x + 20$. The value of p at $x = 1$ is $1 - 5 - 4 + 20 = 12$.

32. **(Missile)** A missile hits its target with probability 0.3. Find the least number of missiles to be launched to insure an 80% probability of hitting the target.

[a] 5 (b) 6 (c) 8 (d) 9 (e) None

Solution to 32. The missile misses with probability 0.7, so n missiles miss the target with probability 0.7^n. To find the least n, solve the inequality $1 - 0.7^n > 0.8$ by trial–and–error. The least one is $n = 5$.

33. **(Perimeter)** Find the perimeter of a right triangle having an inscribed circle of diameter 1 and one leg of $7/5$ units.

[a] 6.3 (b) $\frac{19}{3}$ (c) $\frac{59}{9}$ (d) 6 (e) None

Solution to 33. Let $r = 1/2$ be the radius of the circle. Let $a = 7/5$ be the known leg. The circle's contact points divide the triangle's sides into segments of lengths r, x; x, $a-r$; $a-r$, r, where x is unknown. An equation for x is obtained by finding the area in two ways: half the base times the altitude, and half the perimeter times r. This equality is $\frac{1}{2}a(r+x) = (a+x)r$, hence $x = ar/(a-2r)$. The triangle perimeter is $2r+2x+2(a-r) = 2a(a-r)/(a-2r) = 63/10$.

34. **(Shells)** Two spherical surfaces of radii 13 and 15 have centers 14 apart. Find the radius of the circle of intersection.

(a) $10\frac{3}{7}$ (b) $11\frac{4}{7}$ (c) $12\frac{1}{7}$ (d) $12\frac{2}{7}$ [**e**] None

Solution to 34. Let r be the unknown radius. The circle is in a plane at distance x from the center of the smaller sphere, and at distance $14-x$ from the center of the larger sphere. The Pythagorean Theorem implies that $x^2+r^2 = 13^2$ and also $(14-x)^2+r^2 = 15^2$. Subtract these two equations to isolate $x = 140/28 = 5$. Then $r^2 = 13^2 - x^2 = 144$ and $r = 12$.

35. **(Cubic)** Find the sum of the reciprocals of the roots of $3x^3 + 2x^2 - 5x + 2 = 0$.

[**a**] $\frac{5}{2}$ (b) $\frac{5}{3}$ (c) $-\frac{5}{3}$ (d) $\frac{2}{3}$ (e) $-\frac{2}{3}$

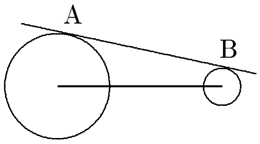
Solution to 35. Replace x by $1/x$ to get $2x^3-5x^2+2x+3 = 0$. Then $-5/2$ equals the negative of the sum of the roots, giving answer $5/2$.

36. **(Ratio)** A parallelogram of altitude 30 and with diagonals 34 and 78 has area A. There is another parallelogram with the same altitude and diagonals having area $B > A$. Find A/B.

[**a**] $\frac{7}{11}$ (b) $\frac{2}{3}$ (c) $\frac{9}{13}$ (d) $\frac{5}{7}$ (e) $\frac{11}{15}$

Solution to 36. Draw a figure for each. One has base 44, the other has base 28. The areas are then *base* $\times$ *altitude*, $A = (28)(30)$ and $B = (44)(30)$. The ratio A/B is $28/44$ or $7/11$.

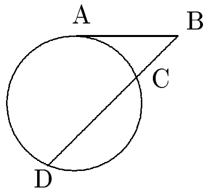
37. **(Yardstick)** A yardstick contacts a large tomato tin and a small tomato paste tin, as in the figure. Find the distance between the contact points, given the tins have diameters 20 and 4, with centers 17 apart.



(a) $\frac{29}{8}\sqrt{17}$ [**b**] 15 (c) $\frac{15}{4}\sqrt{17}$ (d) 16 (e) None

Solution to 37. The contact points are along an external tangent to the two circles. Draw a line parallel to this tangent and passing through the center of the small circle. Complete the sides of a rectangle of dimensions 2 and x, with parallel sides along radii of the two circles. Isolate from this figure a right triangle of sides x, 8, 17. By the Pythagorean Theorem, $x = 15$.

38. **(Tangents)** Given segment $\overline{AB}$ tangent at A, $AB = 9x - 22$, $BC = 7x-18$ and $DC = 2x$, find the sum of all possible solutions x.



(a) $\frac{8}{3}$ (b) 4 (c) $\frac{16}{3}$ [**d**] 6
(e) None

Solution to 38. **Theorem**: *If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment is equal to the product of the measures of the secant segment and its external secant segment.* See Merrill Geometry (1993), p 449. Applying the theorem gives $(9x-22)^2 = (7x-18)(9x-18)$. Taking roots gives $x = 10/3$ and $x = 8/3$. Both are solutions (neither is extraneous). Therefore the sum of the roots is 6.

39. **(Urn)** Urn A has 2 white and 4 red balls; Urn B has 5 white and 3 red. A ball is randomly chosen from urn A, deposited in urn B, then a ball is selected randomly from urn B. Given that a white ball was selected from B, find the probability that the ball transferred from A was white.

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ [**c**] $\frac{3}{8}$ (d) $\frac{4}{9}$ (e) None

Solution to 39. The scheme uses conditional probability calculations, as follows: $P(B_1 = W|B_2 = W) = r/(s+t)$, where $r = P(B_1 = W)P(B_2 = W|B_1 = W)$, $s = P(B_1 = R)P(B_2 = W|B_1 = R)$, $t = P(B_1 = W)P(B_2 = W|B_1 = W)$. Then $r = t = (1/3)(2/3)$ and $s = (2/3)(5/9)$. The probability is $r/(s+t) = 3/8$.

40. **(Median)** A scalene triangle of area $15\sqrt{11}$ has medians 9 and 15. Determine the unknown median.

(a) 9 (b) 24 (c) $6\sqrt{15}$ [**d**] $3\sqrt{59}$ (e) $3\sqrt{57}$

Solution to 40. Let M be the intersection point of the medians in triangle ABC. Construct a second triangle ACD. Label sides $AD = 6$, $DC = 10$ and draw the medians through M, with segment lengths 3, 6, 5, 10, y, $2y$, with y unknown. Then DM is also divided into segments of lengths y and y. The area of triangle AMC is 1/3 of the area of triangle ABC, making $5\sqrt{11}$. Apply Heron's area formula for triangles to obtain the second equation $5\sqrt{11} = \sqrt{(y-2)(y+2)(8-y)(8+y)}$. This produces the clever equation $(y^2-59)(y^2-9) = 0$ for unknown y. The unknown median has length $3y$, so 9 and $3\sqrt{59}$ are the candidates. The value 9 is impossible because the triangle is scalene.

1996 Utah State Math Contest Answer Keys

Answers to 1996 Exam Grade 7

1. (a)	11. (a)	21. (a)	31. (a)
2. (d)	12. (a)	22. (a)	32. (b)
3. (e)	13. (d)	23. (e)	33. (a)
4. (a)	14. (a)	24. (a)	34. (d)
5. (d)	15. (d)	25. (c)	35. (c)
6. (b)	16. (c)	26. (d)	36. (d)
7. (b)	17. (c)	27. (b)	37. (e)
8. (c)	18. (b)	28. (a)	38. (b)
9. (b)	19. (c)	29. (c)	39. (d)
10. (c)	20. (b)	30. (c)	40. (a)

Answers to 1996 Exam Grade 8

1. (a)	11. (e)	21. (b)	31. (a)
2. (d)	12. (e)	22. (b)	32. (b)
3. (a)	13. (b)	23. (d)	33. (a)
4. (d)	14. (c)	24. (c)	34. (d)
5. (c)	15. (c)	25. (c)	35. (c)
6. (d)	16. (c)	26. (c)	36. (d)
7. (a)	17. (a)	27. (d)	37. (e)
8. (b)	18. (a)	28. (d)	38. (b)
9. (e)	19. (c)	29. (d)	39. (d)
10. (b)	20. (e)	30. (d)	40. (a)

Answers to 1996 Exam Grade 9

1. (b)	11. (c)	21. (c)	31. (a)
2. (c)	12. (a)	22. (c)	32. (b)
3. (b)	13. (c)	23. (d)	33. (a)
4. (e)	14. (a)	24. (c)	34. (d)
5. (d)	15. (b)	25. (e)	35. (c)
6. (d)	16. (d)	26. (c)	36. (d)
7. (c)	17. (b)	27. (b)	37. (e)
8. (b)	18. (d)	28. (d)	38. (b)
9. (a)	19. (c)	29. (b)	39. (d)
10. (e)	20. (a)	30. (a)	40. (a)

Answers to 1996 Exam Grade 10

1. (b)	11. (e)	21. (d)	31. (c)
2. (c)	12. (c)	22. (c)	32. (a)
3. (e)	13. (d)	23. (e)	33. (a)
4. (d)	14. (c)	24. (a)	34. (e)
5. (d)	15. (d)	25. (a)	35. (a)
6. (d)	16. (a)	26. (c)	36. (a)
7. (c)	17. (a)	27. (a)	37. (b)
8. (b)	18. (c)	28. (a)	38. (d)
9. (b)	19. (d)	29. (d)	39. (c)
10. (d)	20. (e)	30. (e)	40. (d)

Answers to 1996 Exam Grade 11

1. (d)	11. (d)	21. (c)	31. (c)
2. (b)	12. (d)	22. (d)	32. (a)
3. (d)	13. (e)	23. (e)	33. (a)
4. (d)	14. (c)	24. (d)	34. (e)
5. (a)	15. (b)	25. (d)	35. (a)
6. (e)	16. (e)	26. (c)	36. (a)
7. (a)	17. (c)	27. (c)	37. (b)
8. (a)	18. (e)	28. (a)	38. (d)
9. (e)	19. (a)	29. (c)	39. (c)
10. (a)	20. (b)	30. (b)	40. (d)

Answers to 1996 Exam Grade 12

1. (a)	11. (b)	21. (c)	31. (c)
2. (a)	12. (a)	22. (b)	32. (a)
3. (c)	13. (c)	23. (a)	33. (a)
4. (d)	14. (c)	24. (c)	34. (e)
5. (d)	15. (d)	25. (b)	35. (a)
6. (b)	16. (d)	26. (d)	36. (a)
7. (a)	17. (d)	27. (c)	37. (b)
8. (e)	18. (c)	28. (e)	38. (d)
9. (a)	19. (c)	29. (d)	39. (c)
10. (d)	20. (a)	30. (a)	40. (d)