1995 Utah State Math Contest
University of Utah
March 17, 1995
Teacher's Solution Set

1995 Junior Exam Grade 7 with Key and Solutions

Answers to 1995 Exam Grade 7			
1. **(d)**	11. **(a)**	21. **(b)**	31. **(a)**
2. **(b)**	12. **(b)**	22. **(a)**	32. **(a)**
3. **(b)**	13. **(c)**	23. **(a)**	33. **(d)**
4. **(a)**	14. **(d)**	24. **(c)**	34. **(b)**
5. **(d)**	15. **(c)**	25. **(d)**	35. **(d)**
6. **(b)**	16. **(c)**	26. **(a)**	36. **(d)**
7. **(e)**	17. **(d)**	27. **(c)**	37. **(e)**
8. **(b)**	18. **(c)**	28. **(d)**	38. **(c)**
9. **(a)**	19. **(b)**	29. **(b)**	39. **(d)**
10. **(d)**	20. **(c)**	30. **(d)**	40. **(c)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Quotient)** Evaluate $\dfrac{-(|-13|-|-3|)}{\frac{81}{7}-\frac{22}{14}}$.

(a) 1.6 (b) -1.6 (c) 1 [**d**] -1 (e) None

Solution to 1. The top of the fraction is $|-13|-|-3| = 13-3 = 10$. The bottom of the fraction is $81/7 - 22/14 = 81/7 - 11/7 = 10$. Because of the leading minus sign, the answer is $-10/10 = -1$.

2. **(Tank)** A gas tank becomes half full after adding 2.5 gallons to a tank 1/3 full. Find the tank capacity in gallons.

(a) 16 [**b**] 15 (c) 14 (d) 12 (e) 11

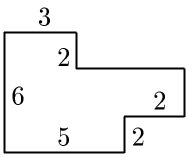
Solution to 2. Let x be the capacity. Then $x/3 + 2.5 = x/2$. Solving for x gives $x = 15$.

3. **(Sound)** Sound travels in air about 344 meters per second. Find the approximate distance in meters to a lightning flash, given the thunder was heard 4 seconds later.

(a) 1300 [**b**] 1400 (c) 1500 (d) 1600 (e) 1700

Solution to 3. The formula $D = RT$ is applied with $R = 344$ and $T = 4$ to give $D = 1376$, or about 1400 kilometers.

4. **(Area)** Find the area of the quadrilateral in the figure. All angles are right angles.



[**a**] 30 (b) 31 (c) 32 (d) 33 (e) 34

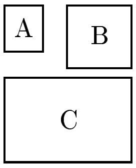
Solution to 4. Break the figure into rectangles and apply the area formula: $area = length \times width$.

5. **(Integers)** Find the greatest integer whose product with -4 is greater than -24.

(a) 11 (b) 1 (c) 3 [**d**] 5 (e) 7

Solution to 5. Let x be the integer. Then $-4x > -24$ implies $-x > -6$ or $x < 6$. The greatest integer is $x = 5$.

6. **(Area)** In the figure, let rectangle C be six times the area of rectangle A, and let rectangle B have area 48 units, which is 12 units more than the area of rectangle A. Find the area of rectangle C.



(a) 256 [**b**] 216 (c) 204 (d) 196 (e) 180

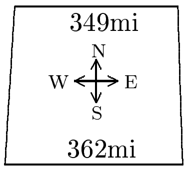
Solution to 6. Use lower case letters for areas. Then $c = 6a$, $b = 48$, $b = 12 + a$. Substituting gives $48 = 12 + a$ or $a = 36$. Then $c = 6a = 6(36) = 216$.

7. **(Mode)** Find the mode for the basketball scores 7, 9, 12, 17, 4, 16, 8, 22, 13.

(a) 9 (b) 12 (c) 12.5 (d) 13 [**e**] None

Solution to 7. There is no mode because each score occurs just once.

8. **(Wyoming)** Estimate the area in square miles of the state of Wyoming in the figure, assuming it has the shape of a trapezoid, with parallel north and south borders 275 miles apart. Units are miles.



(a) 97700 [**b**] 97800 (c) 97900 (d) 98000 (e) 98100

Solution to 8. Multiply the sum of the bases by the altitude and divide by two. The area is $(349 + 362)(275)(1/2) = 97762.5$. The actual area of Wyoming is 97809.

9. **(Sequence)** What is the eighth term divided by the ninth term of the sequence $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$?

[**a**] $\dfrac{80}{81}$ (b) $\dfrac{8}{10}$ (c) $\dfrac{9}{10}$ (d) $\dfrac{17}{19}$ (e) $\dfrac{56}{90}$

Solution to 9. The eighth term is $\dfrac{8}{9}$ and the ninth term is $\dfrac{9}{10}$. Their quotient is $\frac{80}{81}$.

10. **(Apples)** Find the percentage of good apples in a basket of 65 apples containing thirteen rotten apples.

(a) 20% (b) 52% (c) 72% [**d**] 80% (e) None

Solution to 10. The percentage of good apples corresponds to $(65 - 13)/65 = 0.8$ or 80%.

11. **(Divisor)** Which number below divides 1337456631234376?

[**a**] 8 (b) 41 (c) 251 (d) 1973 (e) None

Solution to 11. Let $x = 1337456631234376 = 1000(1337456631234) + 376$. Since 1000 is divisible by 8 and 376 is divisible by 8, then so is x. Most graphing calculators in 1995 supply no real help on the problem, due to display limitations.

12. **(Fractions)** Find a fraction equivalent to $\dfrac{40p^2q}{52pq^2}$.

(a) $\dfrac{p}{q}$ [**b**] $\dfrac{10p}{13q}$ (c) $\dfrac{10p}{q}$ (d) $\dfrac{10}{13}$ (e) None

Solution to 12. Write as $\dfrac{(2)(4)(5)ppq}{(4)(13)pqq}$. Cancel like terms to obtain $\dfrac{(2)(5)p}{13q}$ which equals $\dfrac{10p}{13q}$.

13. **(LCM)** Find the least common multiple of 45, 30, 105.

(a) 5 (b) 315 [**c**] 630 (d) 15 (e) None

Solution to 13. The LCM of three numbers is the smallest number N divisible by each of the three numbers. Write $45 = (3)(3)(5)$, $30 = (2)(3)(5)$, $105 = (5)(3)(7)$. Then $LCM = (3)(3)(5)(2)(7) = 630$.

14. **(Wrench)** A bolt measures about 15/32 inches. Which wrench size in inches below is the best choice to fit the bolt?

(a) 5/8 (b) 11/16 (c) 7/16 [**d**] 1/2 (e) 9/16

Solution to 14. The various values x given in the answers have to satisfy $x > 15/32$ in order to qualify, however, the right answer is the smallest such x. The tests are $15/32 < 5/8$, $15/32 < 11/16$, $15/32 < 7/16$, $15/32 < 1/2$, $15/32 < 9/16$. Only $15/32 < 7/16$ fails. The smallest qualifying size is $x = 1/2$.

15. **(Marbles)** A bag of marbles has 3 blue, 6 red, 2 green and 1 black. Find the probability of randomly drawing a green or black marble.

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ [**c**] $\frac{1}{4}$ (d) $\frac{1}{3}$ (e) None

Solution to 15. The sample set consists of 12 items and 3 of these represent success, so the probability is 3/12 or 1/4.

16. **(Spinners)** Two twelve-hour clock faces are equipped with spinners that always land on a whole number 1 through 12. After spinning each, what is the probability that the sum of the two numbers is greater than 8?

(a) $\frac{23}{36}$ (b) $\frac{25}{36}$ [**c**] $\frac{29}{36}$ (d) $\frac{31}{36}$ (e) None

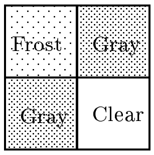
Solution to 16. The spins result in possible events (a, b) where a and b are integers 1–12. Altogether there are $12^2 = 144$ events. An event is successful if $a + b > 8$. These events are $(1, 8)$, $(1, 9)$, $(1, 10)$, $(1, 11)$, $(1, 12)$, $(2, 7)$, $(2, 8)$, $\ldots$, $(2, 12)$, $(3, 6)$, $(3, 7)$, $\ldots$, $(3, 12)$ and so on. Arranging these in a neat table and counting gives the success count as $5 + 6 + 7 + 8 + 9 + 10 + 11$ plus $5(12)$ which is $56 + 60 = 116$. The probability is $116/144$.

17. **(DNA strand)** How many DNA strands can be laid end-to-end across the diameter of a spherical human blood cell of volume $\frac{4}{3}\pi(2^{-17})^3$m? Use 2^{-18}m for the length of a strand.

(a) 1 (b) 2 (c) 3 [**d**] 4 (e) 5

Solution to 17. The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$. The human blood cell has diameter $2R = 2^{-16}$. Divide by $S = 2^{-18}$ to find the number of strands across a diameter: $2R/S = 2^{-16}/2^{-18} = 2^2 = 4$.

18. **(Combinatorics)** How many different square 4-panel window designs can be made using two *Gray* panes, one *Frosted* pane and one *Clear* pane?



(a) 24 (b) 18 [**c**] 12 (d) 10 (e) 4

Solution to 18. The patterns labeled the same way we read a page of text are FGGC, FGCG, FCGG, GFCG, GFGC, GGCF, GGFC, GCFG, GCGF, CFGG, CGFG, CGGF.

19. **(Discount)** Boots priced at \$42 were increased in price by 25% and later sold at a 25% off sale. What was the final sales price?

(a) \$42 [**b**] \$39.38 (c) \$52.50 (d) \$38 (e) \$39

Solution to 19. The price increase was $(0.25)(42) = 10.50$ making a new price of $42 + 10.50 = 52.50$. The 25% off sale uses a sale price of $(0.75)(52.50) = 39.375$ or \$39.38.

20. **(Proportions)** Suppose 9 gallons cost \$12.15. Find x, if x quarts cost \$2.25.

(a) 6 (b) $\frac{19}{3}$ [**c**] $\frac{20}{3}$ (d) 7 (e) $\frac{5}{3}$ None

Solution to 20. A gallon is 4 quarts. So $(4)(9) = 36$ quarts cost 12.15. The proportion is $\dfrac{36}{12.15} = \dfrac{x}{2.25}$. The answer is $x = \dfrac{20}{3}$.

21. **(Linear equation)** Solve for x: $2(x - 1) + 5 = 3(x - 1)$.

(a) 5 [**b**] 6 (c) -5 (d) -6 (e) None

Solution to 21. Swap sides, add $-2(x - 1)$ to each side. Then $3(x - 1) - 2(x - 1) = 5$. The result is $x - 1 = 5$ which gives $x = 6$.

22. **(Squares)** The sum of the squares of two positive integers is 289. What is their sum?

[**a**] 23 (b) 22 (c) 21 (d) 20 (e) None

Solution to 22. This triangle is supposed to be a familiar one of sides 8, 15, 17. If so, then the problem is immediately solved as $8 + 15 = 23$.

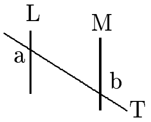
To decide upon the answer without such knowledge, taking into account all possible combinations, let a and b be the integers satisfying $a^2 + b^2 = 289 = 17^2$. Then a and b are the legs of a right triangle, which according to the triangle inequality must each be less than 17. There are only a finite number of possibilities to try: $1 \le a \le 16$, $b = \sqrt{289 - a^2}$. The possibilities are tried: $a = 1$, $b = 12\sqrt{2}$, ..., $a = 16$, $b = \sqrt{33}$. The only integer answers are $a = 8$, $b = 15$ and $a = 15$, $b = 8$. Then $a + b = 23$.

23. **(Complement)** The measure of an angle is 26° more than three times the measure of its complement. Find the measure of the complement.

[**a**] 16° (b) 32° (c) 38.5° (d) 74° (e) None

Solution to 23. Let x be the complement. Then $90 - x$ is the angle and the text implies $90 - x = 26 + 3x$. Solving gives $x = 16$.

24. **(Transversal)** Let lines L and M be parallel and cut by the transversal T as in the figure, forming 8 angles. Given $a = 7x$ and $b = 2x + 75$, in degrees, find the measure in degrees of the largest of the 8 angles.



(a) 15° (b) 75° [**c**] 105° (d) 120° (e) 125°

Solution to 24. A transversal cuts parallel lines in equal angles, therefore, $7x = 2x + 75$. Solving gives $x = 15$ and the two angles are $7x = 105$ and its supplement $180 - 105 = 75$. The largest is 105.

25. **(Area)** Which dimension below is a side of a rectangle of perimeter 16 and area 12?

(a) 3 (b) 4 (c) 5 [**d**] 6 (e) 8

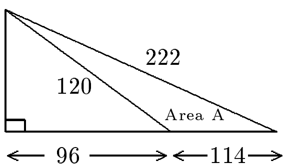
Solution to 25. Draw a figure and try each dimension in the list (a) – (e). The only rectangle with perimeter 16 and area 12 has dimensions 2 and 6.

26. **(Systems)** Find the value of x in the linear system $x + y = 4$, $y - 3x = 8$.

[**a**] -1 (b) 0 (c) 2 (d) 4 (e) 5

Solution to 26. Solve the first equation for x in terms of y: $x = 4 - y$. Put the answer into the second equation $y - 3x = 8$ to get $y - 3(4 - y) = 8$. Then $4y = 12 + 8$ and $y = 5$. Finally, $x = -1$, by using the equation $x + y = 4$ with the value $y = 5$. Faster method: subtract the equations to get $(x + y) - (y - 3x) = 4 - 8$. This reduces to $4x = -4$ or $x = -1$.

27. **(Triangles)** Find area A.
(a) 4100 (b) 4102
[**c**] 4104 (d) 4110
(e) 4116



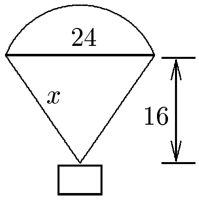
Solution to 27. The Pythagorean theorem applied to the left triangle reveals the altitude x satisfies $x^2 + 96^2 = 120^2$ or $x = 72$. The base is 114. The area is half the base times the altitude, or $(1/2)(114)(72) = 4104$.

28. **(Inclined plane)** Find the slope of an inclined plane which raises an object 1m vertically for each 3m of travel on the inclined surface.

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{3}{2\sqrt{2}}$ [**d**] $\dfrac{\sqrt{2}}{4}$ (e) None

Solution to 28. Form a right triangle with hypotenuse 3 and legs 1, x. Then $x^2 + 1^2 = 3^2$, giving $x = 2\sqrt{2}$. The slope is *rise/run*, which is $1/x = 1/(2\sqrt{2}) = \sqrt{2}/4$.

29. **(Supply drop)** The load of supplies on a parachute hangs 16ft below the canopy, which has diameter 24ft. Find the measure x of a suspension line in feet.



(a) $8\sqrt{13}$ [**b**] 20 (c) $\frac{59}{3}$ (d) 21 (e) $4\sqrt{13}$

Solution to 29. Let x be the length of the suspension line. Then $x^2 = 12^2 + 16^2$ by the Pythagorean theorem. The answer is $x = 20$.

30. **(Perimeter)** Find the perimeter of an equilateral triangle of area $25\sqrt{3}$.

(a) 10 (b) 20 (c) 25 [**d**] 30 (e) None

Solution to 30. Let x be a side of the triangle. Cut the triangle in half to give two equal triangles with sides in the ratio $\frac{1}{2}$:$\frac{\sqrt{3}}{2}$:1. The altitude is $\sqrt{3}x/2$ and the base is x, so the triangle's area is $(1/2)(x)(\sqrt{3}x/2)$, which also equals $25\sqrt{3}$. Solving $(1/2)(x)(\sqrt{3}x/2) = 25\sqrt{3}$ gives $x = 10$. The perimeter is $3x = 30$.

See *infra* for problems 31-40.

1995 Junior Exam Grade 8 with Key and Solutions

Answers to 1995 Exam Grade 8

1. **(c)**	11. **(c)**	21. **(d)**	31. **(a)**
2. **(e)**	12. **(e)**	22. **(c)**	32. **(a)**
3. **(b)**	13. **(d)**	23. **(b)**	33. **(d)**
4. **(e)**	14. **(c)**	24. **(c)**	34. **(b)**
5. **(e)**	15. **(a)**	25. **(c)**	35. **(d)**
6. **(d)**	16. **(c)**	26. **(b)**	36. **(d)**
7. **(c)**	17. **(c)**	27. **(a)**	37. **(e)**
8. **(b)**	18. **(a)**	28. **(d)**	38. **(c)**
9. **(b)**	19. **(b)**	29. **(c)**	39. **(d)**
10. **(b)**	20. **(b)**	30. **(d)**	40. **(c)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Travel)** Jack travels 7 miles the first day and 51 miles the last day, increasing his daily mileage 4 miles per day. Find his total mileage.

   (a) 336 (b) 340 [**c**] 348 (d) 352 (e) 356

   Solution to 1. The trip is 12 days, with daily mileages $7, 11, 15, \ldots, 51$. The sum of these mileages is 348.

2. **(Inequality)** Given $|5-3x| \geq 1$, which of the following is necessarily true?

   (a) $x \geq 2$ (b) $x = 2$ (c) $x < 2$ (d) $x > 4/3$ [**e**] None

   Solution to 2. The inequality means $5 - 3x \geq 1$ or $-(5-3x) \geq 1$. On the number line, these options describe two disjoint intervals. None of the answers supplied describe two intervals. The answer is *None.*

3. **(Volume)** Find the volume in cubic units of a parallelepiped (box) of sides yz^4, yz^2 and y^2z units.

   (a) y^4z^9 [**b**] $(yz)^4z^3$ (c) y^4z^6 (d) $\frac{4}{3}y^4z^7$ (e) None

   Solution to 3. The volume of a box is the product of the side dimensions, which is $yz^4yz^2y^2z = y^4z^7$.

4. **(Powers)** Given $a > 0$, simplify $\dfrac{(a^{x+2})^2}{(a^{x-3})^2(-a)^5}$.

   (a) a^6 (b) $-a^6$ (c) $-a^{-3}$ (d) a^5 [**e**] $-a^5$

   Solution to 4. The fraction numerator is a^{2x+4} and the denominator is $(-1)^5a^{2x-6+5}$. The first reduction is $(-1)^5 = -1$. Powers combine to make $a^{2x+4-2x+6-5}$, giving the simplification $-a^5$.

5. **(Fractional equation)** Solve $\dfrac{3}{x+1} = \dfrac{1}{x+1} - 7$.

   (a) $\frac{-7}{9}$ (b) $\frac{11}{7}$ (c) $\frac{-11}{7}$ (d) $\frac{9}{7}$ [**e**] $\frac{-9}{7}$

   Solution to 5. Combine to $2/(x+1) = -7$, then take reciprocals to get $(x+1)/2 = -1/7$. Solving as a linear equation, $x = -1 - 2/7 = -9/7$.

6. **(Percent)** How many elevenths are equivalent to the fraction of 75 percent?

   (a) $\frac{3}{4}$ (b) $\frac{9}{11}$ (c) $\frac{12}{11}$ [**d**] $8\frac{1}{4}$ (e) None

   Solution to 6. The equation to be solved is $x/11 = 0.75$, which has solution 8.25.

7. **(Line)** Determine the value of r such that the line through the points $(4, r)$ and $(r, 2)$ has slope $-5/3$.

   (a) -7 (b) 1 [**c**] 7 (d) -1 (e) None

   Solution to 7. The slope formula $m = (y_2 - y_1)/(x_2 - x_1)$ applies to give the equation $-5/3 = (2-r)/(r-4)$. Clearing fractions gives $-5(r-4) = 3(2-r)$ which can be solved for $r = 7$.

8. **(Midpoint)** Let P be the midpoint of the segment from $A(8, 4)$ to $B(12, 12)$. Find the coordinates of P.

   (a) $(8, 10)$ [**b**] $(10, 8)$ (c) $(20, 16)$ (d) $(16, 20)$ (e) $(4, 8)$

   Solution to 8. The midpoint (x, y) is given by $x = (8+12)/2$, $y = (4+12)/2$.

9. **(Region)** Find the point (x, y) below which is above the curve $y = |x|$ and below the line $2y + x = 3$.

   (a) $(1.5, 1)$ [**b**] $(-1, 1.5)$ (c) $(1.5, -1)$ (d) $(-1, 0.75)$ (e) None

   Solution to 9. The region above the curve $y = |x|$ is $y > |x|$. The only candidate is $(-1, 1.5)$. The region below the line $2y+x = 3$ is $y < (3-x)/2$. Again, $(-1, 1.5)$ satisfies $1.5 < (3-(-1))/2$. The answer is $(-1, 1.5)$.

10. **(Length)** The length of a rectangle of perimeter 50 cm exceeds its width by 7 cm. Find the length in centimeters.

    (a) 28.5 [**b**] 16 (c) 14 (d) 12 (e) 9

    Solution to 10. The equations are $x = y + 7$, $2x + 2y = 50$. Solving gives $x = 16$, $y = 9$.

11. **(Radicals)** Simplify $\sqrt{(x-1)^2}$.

    (a) $x - 1$ (b) $1 - x$ [**c**] $|1 - x|$ (d) $-|1 - x|$ (e) None

    Solution to 11. The square root is always positive or zero, by convention. The identity $\sqrt{u^2} = |u|$ is applied to give the answer $|x-1|$ which is also $|1-x|$.

12. **(Radicals)** Solve $\sqrt{3x - 14} + x = 6$ for x.

    (a) -5 (b) 4 (c) 6 (d) 10 [**e**] None

    Solution to 12. Put x on the right side of the equation and square both sides to obtain $3x - 14 = 36 - 12x + x^2$. Place all terms on the right and factor with the FOIL method into $0 = (x-5)(x-10)$. The answer 5 is correct. The number 10 is an *extraneous solution*, it does not satisfy the equation.

13. **(Distance)** The distance between points $(-a, 11)$ and $(-3, 5)$ is 10. Find $a > 0$.

    (a) 3 (b) 5 (c) 10 [**d**] 11 (e) None

    Solution to 13. Solve $(-a+3)^2 + (11-5)^2 = 10^2$ for $a > 0$. The possible answers for a are 11 and -5, but only one is positive.

14. **(Cube)** A white plastic cube is painted black and cut into 64 unit cubes. Find the number of unit cubes with exactly two black faces.

    (a) 12 (b) 16 [**c**] 24 (d) 32 (e) None

    Solution to 14. There are twelve edges on the original cube, each of which produced two unit cubes having exactly two black faces. No other unit cubes have two black faces.

15. **(Probability)** In a survey of 700 people concerning two ice cream flavors, 415 liked chocolate, 269 liked vanilla and among these, 124 liked both flavors. Estimate the probability that a person dislikes both flavors.

[**a**] 0.20 (b) 0.22 (c) 0.24 (d) 0.26 (e) 0.28

Solution to 15. The total who like one or the other or both is 415+269 less the number 124 counted twice, or 560. The total disliking both is not 16, but $700 - 560 = 140$. The probability of disliking both is $140/700 = 1/5 = 0.2$.

16. **(Median)** Use the set of data 12, 24, 18, 22, 17, 10. Find the median of all values greater than the median of the whole set.

(a) 12.5 (b) 18 [**c**] 22 (d) 24 (e) None

Solution to 16. The median divides the data set into two halves. From the sorted order 10, 12, 17, 18, 22, 24 the median of the whole set is $(17+18)/2$. The data set greater than this value is 18, 22, 24, which has median 22.

17. **(Quotients)** Simplify $\dfrac{x^4 - 16}{x^4 - 8x^2 + 16}$.

(a) -1 (b) $\frac{x+2}{x-2}$ [**c**] $\frac{x^2+4}{x^2-4}$ (d) $\frac{x^2}{x^2-8}$ (e) None

Solution to 17. The factorizations $x^4 - 16 = (x^2 - 4)(x^2 + 4)$ and $x^4 - 8x^2 + 16 = (x^2 - 4)^2$ are used to obtain $(x^2 + 4)/(x^2 - 4)$.

18. **(Quotient)** Simplify $\left(1 - \dfrac{1}{1+x}\right)^{-1} - \left(\dfrac{1}{1-x} - 1\right)^{-1}$.

[**a**] 2 (b) -2 (c) $\frac{2}{x}$ (d) $\frac{-2}{x}$ (e) None

Solution to 18. The first reciprocal is $\frac{1+x}{x}$. The second is $\frac{1-x}{x}$. Their difference is $2x/x$ or 2.

19. **(Functions)** Given $f(x) = (2x+3)^2$ and $g(y) = \sqrt{y} - 5$, which answer below is a solution x of the equation $g(f(x)) = 8$?

(a) 8 [**b**] -8 (c) 13 (d) -5 (e) None

Solution to 19. The equation is $\sqrt{y} - 5 = 8$ with $y = (2x + 3)^2$. The answer is obtained from $|2x + 3| = 13$ as $2x + 3 = 13$ or $-2x - 3 = 13$. The first gives $x = 5$ while the second gives $x = -8$.

20. **(Trip)** Jack begins a trip at 10:00am, driving 50mph. Jill follows him at 11:30am, going 45mph. When are they 100 miles apart?

(a) 3:30pm [**b**] 4:30pm (c) 4:45pm (d) 5:00pm (e) 5:30pm

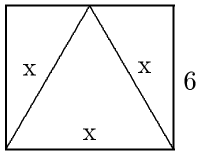
Solution to 20. Jack has traveled 75 miles when Jill starts her trip. Their speed difference is 5mph. The distance between them grows by 5 miles per hour, therefore, after 5 hours an additional 25 miles has accumulated, making $75 + 25 = 100$ miles. Five hours after 11:30am is 4:30pm.

21. **(Chemistry)** Ten liters of a 50% alcohol solution are replaced by 100% solution to make a 75% solution. Find the number of liters in the first solution.

(a) 35 (b) 30 (c) 25 [**d**] 20 (e) 15

Solution to 21. Let x be the number of liters of solution before replacement. Then $0.5(x - 10) + 1.0(10) = 0.75x$, giving the linear equation $2x + 20 = 3x$. Solving, $x = 20$.

22. **(Rectangle)** Find the area in cm^2 of the region inside the rectangle and outside the equilateral triangle of side x. Units are centimeters.



(a) 20 (b) $8\sqrt{6}$ [**c**] $12\sqrt{3}$ (d) 21 (e) $24\sqrt{3}$

Solution to 22. The area of the rectangle is $6x$ and of the triangle $\frac{1}{2}(x)(6)$, so the region of interest has area $6x - 3x$ or $3x$. The value of x is found from the Pythagorean theorem by using sides 6 and $x/2$ with hypotenuse x: $x^2 = 6^2 + (x/2)^2$. Solving gives $x = 4\sqrt{3}$ and then the area requested is $3x = 12\sqrt{3}$.

23. **(Acute)** For which values of d is $(30 - d/2)$ the measure of a positive acute angle in degrees?

(a) $-60 < d < 45$ [**b**] $-120 < d < 60$
(c) $-30 < d < 60$ (d) $|d| < 60$ (e) $|d| < 45$

Solution to 23. The requirement is $0 < 30 - d/2 < 90$, which is equivalent to $-90 < d/2 - 30 < 0$ and finally $-120 < d < 60$.

24. **(Trapezoid)** A trapezoid of area 162 square units has parallel sides of x and $2x + 6$ units. Given these parallel sides are 12 units apart, find x.

(a) 5 (b) 6 [**c**] 7 (d) 8 (e) None

Solution to 24. The average of the parallel sides x and $2x + 6$ times the altitude equals the area, which gives the equation $(x + 2x + 6)(12)/2 = 162$. Solving for x gives $x = 7$.

25. **(Algebra)** Given $R = 2x - 1$, $S = 3x + 2$ and $T = -5x - 3$, find $R(S + T)$.

(a) $4x^2$ (b) 1 [**c**] $1 - 4x^2$ (d) $4x^2 - 1$ (e) None

Solution to 25. First, $S + T = 3x + 2 - 5x - 3 = -2x - 1$. Then $R(S + T) = (2x - 1)(-2x - 1) = (1 - 2x)(1 + 2x) = 1 - 4x^2$.

26. **(Algebra)** Find the coefficient of the term x^3 in the polynomial $(x - 3)^2(2x + 11)(5x - 4)$.

(a) 13 [**b**] -13 (c) 14 (d) -14 (e) -5

Solution to 26. Multiply it out to get $(x-3)^2(2x+11)(5x - 4) = 10x^4 - 13x^3 - 236x^2 + 687x - 396$. The coefficient of x^3 is -13. Alternatively, without multiplying it out, the coefficient is the negative sum of the roots times the coefficient of x^4, or $-(3 + 3 - 11/2 + 4/5)(10) = -13$.

27. **(Algebra)** Let y be the sum of all values of k such that $x^2 + kx + 8 = 0$ has integer roots. Find y.

[**a**] 0 (b) 1 (c) 8 (d) −8 (e) None

Solution to 27. The value $-k$ is the sum of the roots, therefore k is an integer. The rational root theorem says that the possible integer roots are 1, −1, 2, −2, 4, −4, 8, −8. The values of k corresponding to these roots are, respectively, −9, 9, −6, 6, −6, 6, −9, 9. Then $y = -9 + 9 - 6 + 6 = 0$.

28. **(Tile)** Square tiles manufactured in Toledo are stocked in sizes having area $25x^2 - 90x + 81$ square inches, where x is a positive integer less than 15. Find the perimeter in inches of the smallest tile in stock.

(a) 36 (b) 24 (c) 16 [**d**] 4 (e) None

Solution to 28. The area is $(5x-9)^2$. The perimeter is $4|5x-9|$. On the integer domain $0 < x < 15$ the minimum value of the perimeter is 4.

29. **(Rational expression)** Simplify

$$\frac{x^2+5x+6}{x^2-x-12}\,\frac{x-4}{x^2+11x+18} \div \frac{x+7}{x^2+14x+45}.$$

(a) $\frac{5}{7}$ (b) $\frac{x-4}{x+7}$ [**c**] $\frac{x+5}{x+7}$ (d) $\frac{x+9}{x+7}$ (e) $\frac{x+2}{x+7}$

Solution to 29. The factorizations are $x^2+5x+6 = (x+3)(x+2)$, $x^2 - x - 12 = (x-4)(x+3)$, $x^2 + 11x + 18 = (x+9)(x+2)$, $x^2 + 14x + 45 = (x+9)(x+5)$. After inverting and multiplying, the product reduces to $(x+5)/(x+7)$.

30. **(Area)** Find the area of a circle of circumference 22.

(a) 196π (b) 49π (c) 121π [**d**] $121/\pi$ (e) None

Solution to 30. The circumference is $2\pi R$, which means $2\pi R = 22$. The area is $\pi R^2 = \pi(11/\pi))^2 = 121/\pi$.

See *infra* for problems 31-40.

1995 Junior Exam Grade 9 with Key and Solutions

Answers to 1995 Exam Grade 9			
1. **(c)**	11. **(e)**	21. **(c)**	31. **(a)**
2. **(b)**	12. **(c)**	22. **(a)**	32. **(a)**
3. **(c)**	13. **(d)**	23. **(a)**	33. **(d)**
4. **(d)**	14. **(c)**	24. **(a)**	34. **(b)**
5. **(c)**	15. **(b)**	25. **(c)**	35. **(d)**
6. **(a)**	16. **(c)**	26. **(b)**	36. **(d)**
7. **(d)**	17. **(c)**	27. **(c)**	37. **(e)**
8. **(a)**	18. **(d)**	28. **(a)**	38. **(c)**
9. **(e)**	19. **(b)**	29. **(c)**	39. **(d)**
10. **(a)**	20. **(b)**	30. **(e)**	40. **(c)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Jackpot)** A jackpot of \$26.50 is divided with 60¢ to each man, 95¢ to each woman and 70¢ left over. A division of 75¢ per person would be short 50¢. Find the number of women.

(a) 10 (b) 11 [**c**] 12 (d) 13 (e) 14

Solution to 1. Let M be the number of men, W the number of women. Then $75(M+W) = 2650+50$, $60M + 95W = 2650 - 70$. Solving gives $W = 12$ and $M = 24$.

2. **(Variation)** Let z vary inversely with x and directly with y, taking on the value $z = a > 0$ when $y = 2$ and $x = 8/a$. Find y when $z = a^2$ and $x = (2a)^3$.

(a) a^5 [**b**] $2a^5$ (c) $4a^5$ (d) $a^5/2$ (e) None

Solution to 2. The variation has the form $z = cy/x$ for some c. The value of c is known from $z = a$, $x = 8/a$, $y = 2$ by substitution into $z = cy/x$: $a = 2c/(8/a)$ or $c = 4$. Then $z = 4y/x$ gives $y = xz/4$ and at $z = a^2$, $x = (2a)^3$ the answer is $y = (2a)^3a^2/4 = 2a^5$.

3. **(Supplement)** The measure of an angle is one-fourth the measure of its supplement. Find the measure of its complement.

(a) 126° (b) 36° [**c**] 54° (d) 144° (e) None

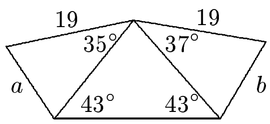
Solution to 3. Let x be the angle. The supplement is $180 - x$. Then $x = (180-x)/4$ gives $5x = 180$ and $x = 36$. The complement is $90 - x$ or 54.

4. **(Pens)** A box with 75 pens has 24% red pens, 17 blue pens and the rest are green. What fraction are green?

(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{7}{15}$ [**d**] $\frac{8}{15}$ (e) None

Solution to 4. The number of red pens is $0.24(75) = 18$, so there are $75 - 17 - 18 = 40$ green pens. The fraction is $40/75$ or $8/15$.

5. **(Inequality)** Decide the correct statement for lengths a and b in the figure.



(a) $a = b$ (b) $a > b$
[**c**] $b > a$ (d) undecidable
(e) None

Solution to 5. The triangle with equal base angles of 43° is isosceles. The lengths a and b are the third sides of two triangles having two congruent sides. The SAS inequality (Hinge Theorem) implies that $a < b$, because the first included angle 35° is less than the second included angle 37°.

6. **(Algebra)** Let $x = a - b$ and $y = a^3 - 3a^2b + 3ab^2 - b^3$. Find the formula valid for all a and b.

[**a**] $y^2 = x^6$ (b) $y = x^4$ (c) $y^3 = x$ (d) $y = -x^3$ (e) None

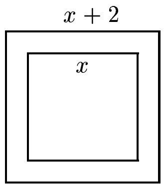
Solution to 6. The expansion of $(a-b)^3$ is y, therefore $y = x^3$ and $y^2 = x^6$.

7. **(Triangle)** A triangle has angles of x, $x+5$ and $2x+3$ degrees. Find the number of degrees in the largest angle.

(a) 43 (b) 48 (c) 79 **[d]** 89 (e) 90

Solution to 7. The angles add to 180, giving the equation $x + x + 5 + 2x + 3 = 180$, having solution $x = 43$. The other angles are $(43) + 5 = 48$ and $2(43) + 3 = 89$. The largest is 89.

8. **(Photo)** A square picture frame is to be designed as in the figure so that $\frac{2}{3}$ of the total area is used by the inner frame. Find the area of the inner frame in square units.



[a] $40 + 16\sqrt{6}$ (b) $40 + 23\sqrt{3}$ (c) $40 + 28\sqrt{2}$
(d) $33 + 19\sqrt{6}$ (e) None

Solution to 8. The inner square has area x^2 and the outer one has area $(x+2)^2$, therefore $\frac{2}{3}(x+2)^2 = x^2$. Solving for $x = 4 + 2\sqrt{6}$ results in inner area $x^2 = 40 + 16\sqrt{6}$.

9. **(Doubling)** A positive number is multiplied by itself and then the new number is doubled to obtain 32. What is the original number?

(a) 16 (b) 12 (c) 10 (d) 8 **[e]** None

Solution to 9. Let x be the number. Then $2x^2 = 32$. To solve $x^2 = 16$, ask "which number x when squared equals 16", to obtain $x = 4$.

10. **(Socks)** Find the probability that two black socks are drawn randomly without replacement from a drawer having 8 black and 5 blue socks.

[a] $\frac{14}{39}$ (b) $\frac{10}{39}$ (c) $\frac{5}{39}$ (d) $\frac{1}{39}$ (e) $\frac{56}{169}$

Solution to 10. The events possible are (a, b) where a and b are colors, black or blue. The event size is $13(12) = 156$. There are $8(7)$ events which choose a and b both black. The probability is $56/156 = 14/39$.

11. **(Functions)** Given $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 4x + 4}$, find $f(-1/2)$.

(a) $\frac{1}{4}$ (b) 9 (c) $\frac{9}{4}$ (d) 5^{-2} **[e]** 3^{-2}

Solution to 11. This is the value of $y = (x^2 + 2x + 1)/(x^2 + 4x + 4)$ when $x = -1/2$. Then $x^2+2x+1 = (x+1)^2 = (1/2)^2 = 1/4$ and $x^2+4x+4 = (x+2)^2 = (3/2)^2 = 9/4$. So $y = (1/4)/(9/4) = 1/9$.

12. **(Equations)** The system $ax + y = 6$, $ax + by = 10$ has the solution $x = -1$, $y = 2$. Find b.

(a) -4 (b) 4 **[c]** 3 (d) -3 (e) 7

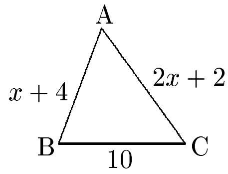
Solution to 12. Substitute $x = -1$ and $y = 2$ to obtain the system $-a + 2 = 6$, $-a + 2b = 10$. Then $a = -4$. Substitute $a = -4$ into $-a + 2b = 10$ to obtain $2b = 6$ or $b = 3$.

13. **(Webbing)** Given seven points in the plane, no three on the same line, find the number of line segments needed to connect each pair of points.

(a) 11 (b) 17 (c) 20 **[d]** 21 (e) 23

Solution to 13. Take a given point of the seven and connect a line segment to each of the other six points. It takes 6 segments. Repeat with a different point, then only $6 - 1$ segments are needed, since the first point was already connected. The pattern: $6 + 5 + 4 + 3 + 2 + 1 = 21$ segments are needed.

14. **(Vertex)** Triangle ABC is isosceles with perimeter between 23 and 32. Locate the vertex angle.
(a) $\angle A$ (b) $\angle B$ **[c]** $\angle C$
(d) undecidable (e) None



Solution to 14. The inequality $23 < 2x + 2 + 10 + x + 4 < 32$ means $\frac{7}{3} < x < \frac{16}{3}$. To be isosceles, one of these must be true: (1) $2x + 2 = 10$, (2) $2x + 2 = x + 4$ or (3) $x + 4 = 10$. The only one that can work is $x = 4$. So $2x + 2 = 10$ and the angle at C is the vertex.

15. **(Clock)** As the hands of a clock move from 3am to 2:59pm, how many times do they form a right angle?

(a) 18 **[b]** 22 (c) 23 (d) 24 (e) 25

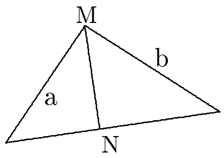
Solution to 15. This problem can be solved by drawing 24 clock diagrams or by winding a standard watch with hour and minute hands. We explain below why the answer is not 24 and not 23.

If the minute hand and the hour hand form a right angle, then a fixed number of minutes a elapses before the hands can form another right angle. A simple estimate is $a = 32.75$, approximately. The fact $a > 30$ is enough to argue that 24 is the wrong answer. As a check on the logic, $32.75(22) > 720$, which accounts for the full number of minutes $30(24) = 720$ from 3am to 3pm.

In the first hour, 3am to 4am, the hands are at right angles 2 times, once at 3am and then after a minutes. During the four hour period from 4am to 7:59am, the hands are at right angles two times per hour for 4 hours, making 8 times. In the time period, 8am to 8:59pm, they are at right angles only once. The total for five hours and 59 minutes is $2 + 8 + 1 = 11$. The geometry from 3am to 8:59am is the same as 9am to 2:59pm. So the total is $11 + 11 = 22$.

A precise solution can be based upon the angle increase between hands, which is 330° per hour. Let t be in hours and let m stand for an integer. For a right angle to occur, $330t$ must be an integral multiple of 180, i.e., $330t = 180m$. The domain of t is $0 \le t \le 11\frac{59}{60}$. Therefore, m satisfies $0 \le m = \frac{330}{180}t \le \frac{330}{180}(11 + \frac{59}{60}) = \frac{7909}{360} < 22$. This implies that m can be any integer from 0 to 21, corresponding to 22 times t.

16. (**Altitude**) In the figure, MN is both a median and an altitude. Given $a = x + 7$ and $b = 2x - 15$, find the unknown x.



(a) 12 (b) 16 [**c**] 22 (d) 44 (e) None

Solution to 16. The triangle must be isosceles, therefore $a = b$ and $2x - 15 = x + 7$. Solving determines $x = 22$.

17. (**Median**) Let right $\triangle ABC$ have legs 12 and 16. Find the length of the shortest median to the nearest tenth.

(a) 6.0 (b) 8.0 [**c**] 10.0 (d) 14.4 (e) 17.0

Solution to 17. Assume the triangle has vertices $(0, 0)$, $(16, 0)$ and $(0, 12)$. Using the midpoint formula, the missing endpoint of each median can be computed. By geometry, the shortest one connects $(0, 0)$ to the midpoint of the hypotenuse, which is at $(8, 6)$. The length of that median is 10, by the distance formula. The other medians have lengths $4\sqrt{13}$ and $2\sqrt{73}$, by application of the same method.

18. (**Cube**) The diagonal of a cube is 96cm. Find the volume in cm^3.

(a) $98294\sqrt{3}$ (b) $98296\sqrt{3}$ (c) $98300\sqrt{3}$
[**d**] $98304\sqrt{3}$ (e) $98306\sqrt{3}$

Solution to 18. The diagonal 96 is the hypotenuse of a right triangle of legs a and b. Let b equal the measure of a side of the cube. Then leg a is the hypotenuse of a right triangle of sides b and b, making $a^2 = b^2 + b^2$. Finally, $96^2 = a^2 + b^2 = 3b^2$ and $b = 32\sqrt{3}$, giving volume $b^3 = \sqrt{27}(2^{15}) = 98304\sqrt{3}$.

19. (**Planes**) How many planes are determined by five points in space, no four of which are coplanar?

(a) 8 [**b**] 10 (c) 11 (d) 14 (e) None

Solution to 19. Each three set of points determines a plane, but taking all combinations leads to duplicate planes. The distinct ones number 10 in all.

20. (**Inequality**) Solve $|x - 2| + |x + 2| \le 4$.

(a) $|x| \le 1$ [**b**] $|x| \le 2$ (c) $-1 \le x \le 2$
(d) $-2 \le x \le 1$ (e) None

Solution to 20. Since $|x - 2| = \pm(x - 2)$ and $|x + 2| = \pm(x + 2)$ the left side of the inequality stands for one of $(x-2)+(x+2)$, $-(x-2)+(x+2)$, $(x - 2) - (x + 2)$ or $-(x - 2) - (x + 2)$. The cases are $2x \le 4$, $4 \le 4$, $-4 \le 4$ or $-2x \le 4$. The only conditions on x are $-2 \le x \le 2$, or $|x| \le 2$.

21. (**Line**) Determine the value of r such that the line through the points $(4, r)$ and $(r, 2)$ has slope $-5/3$.

(a) -7 (b) 1 [**c**] 7 (d) -1 (e) None

Solution to 21. The slope formula $m = (y_2 - y_1)/(x_2 - x_1)$ applies to give the equation $-5/3 = (2 - r)/(r - 4)$. Clearing fractions gives $-5(r - 4) = 3(2 - r)$ which can be solved for $r = 7$.

22. (**Parallelogram**) Three vertices of a parallelogram are $(10, 3)$, $(-1, 2)$ and $(1, -1)$. Which point below is a possible fourth vertex?

[**a**] $(-10, -2)$ (b) $(12, 2)$ (c) $(8, 5)$ (d) $(7, 6)$
(e) None

Solution to 22. Draw a diagram. Only one seems worth testing, namely $(-10, -2)$. Opposite sides have the same slope, so it's a parallelogram.

23. (**Triangle**) Classify the triangle ABC with vertices $A(2, -4)$, $B(-3, 1)$ and $C(1, 6)$.

[**a**] obtuse scalene (b) acute scalene
(c) right scalene (d) isosceles (e) None

Solution to 23. By the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ the sides have lengths $\sqrt{50}$, $\sqrt{41}$ and $\sqrt{101}$, so it's scalene. The angle at B is not a right angle because the Pythagorean relation fails: $\sqrt{50}^2 + \sqrt{41}^2 \ne \sqrt{101}^2$. In fact, this relation says that the angle at B is obtuse.

24. (**Bacteria**) A population of bacteria triples each day, reaching a size of 2187000 in 144 hours. Find the initial population.

[**a**] 3000 (b) 2500 (c) 2000 (d) 1000 (e) None

Solution to 24. Let x be the initial population size. Then after n hours the population is $3^{n/24}(x)$. After 144 hours, $n/24 = 144/24 = 6$ and $3^6 x = 2187000$. Dividing by 3 six times into 2187000 gives 3000.

25. (**Remainder**) What is the remainder when 3^{251} is divided by 5?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) 0

Solution to 25. The first few powers are 3, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$. The remainders after division by 5 are 3, 4, 2, 1, 3. The pattern repeats, each four causing the remainders to repeat. Since $251/4 = 62\frac{3}{4}$, the remainder is the third answer in the list of four repeated values, namely, 2.

26. (**Trapezoid**) Find the length of the median of a trapezoid with vertices $(0, 0)$, $(0, 3)$, $(4, 4)$ and $(4, -1)$.

(a) 5 [**b**] 4 (c) 3 (d) 2 (e) None

Solution to 26. The parallel bases have lengths 3 and 5, by the distance formula. The median length is half of their sum, $(3+5)/2 = 4$. It can also be worked using the midpoint formula and a suitable diagram.

27. **(Paint)** Two rectangular solids are similar with the ratios between corresponding sides at 3:1. The smaller requires x gallons to paint its surface. Find the number of gallons required to paint the surface of the larger.

(a) x^2 (b) $3x$ [**c**] $9x$ (d) $27x$ (e) None

Solution to 27. The dimensions a, b, c of the smaller correspond to dimensions $3a$, $3b$, $3c$ of the larger. Therefore, the surface area of the larger solid is 9 times larger than the surface area of the smaller solid. The paint requirement is $9x$ gallons.

28. **(Series)** Sum the series $\frac{1}{2} + (\frac{1}{3} + \frac{2}{3}) + (\frac{1}{4} + \frac{2}{4} + \frac{3}{4}) + (\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}) + \cdots + (\frac{1}{100} + \frac{2}{100} + \frac{3}{100} + \cdots + \frac{99}{100})$.

[**a**] 2475 (b) 4950 (c) 2476 (d) 4952 (e) None

Solution to 28. The terms in parentheses can be simplified by using the formula $1 + 2 + 3 + \cdots + n = n(n+1)/2$ (Gauss' childhood sum formula). The simplifications are $\frac{1}{2}$, $\frac{2(3)/2}{3}$, $\frac{3(4)/2}{4}$, $\frac{4(5)/2}{5}$, $\ldots$, $\frac{99(100)/2}{100}$. This is half of $1+2+3+4+\cdots+99 = 99(99+1)/2 = 4950$. The series sums to 2475.

29. **(Polynomials)** Let p be a cubic polynomial in x with roots -2, 2 and 5. Assume $p = 20$ when $x = 0$. Find p when $x = 1$.

(a) 10 (b) 11 [**c**] 12 (d) 13 (e) 14

Solution to 29. By the factor theorem, $(x + 2)$, $(x - 2)$ and $(x - 5)$ are factors of p. The polynomial p must be a constant multiple of $(x + 2)(x - 2)(x - 5)$. Briefly, this means $p = c(x + 2)(x - 2)(x - 5)$ for some constant c. In expanded form, $p = c(x^3 - 5x^2 - 4x + 20)$. Since $p = 20$ at $x = 0$, then $c = 1$ and $p = x^3 - 5x^2 - 4x + 20$. The value of p at $x = 1$ is $1 - 5 - 4 + 20 = 12$.

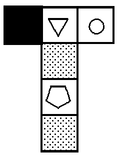
30. **(Rectangle)** A rectangle has a perimeter of 42cm. Which of the following could be its area in cm^2?

(a) 125 (b) 121 (c) 120 (d) 116 [**e**] 110

Solution to 30. Let a and b be its dimensions. Then $2a + 2b = 42$, making $a + b = 21$. The area is $ab = a(21 - a)$. The largest positive value of the quadratic $y = x(21 - x)$ occurs at its vertex, at $x = 21/2$, $y = 441/4$. No rectangle can have area larger than $441/4$ and perimeter 42. This eliminates all but one of the answers. To obtain a rectangle of area 110, solve $110 = x(21 - x)$ for $x = 10$, $x = 11$. Possible rectangles have dimensions 10, 11.

Common Problems 31-40

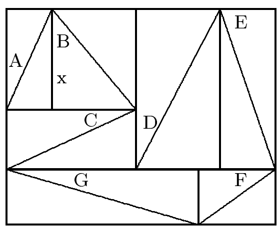
31. **(Probability)** A moth lands on some square in the figure, then randomly moves just once horizontally or vertically to an adjacent square. Find the probability that the moth ends up on the square with the pentagon.



[**a**] $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{24}$ (d) $\frac{1}{8}$ (e) None

Solution to 31. Label the squares as 1 through 6, numbering 1, 2, 3 across the top of the "T" (2 on the triangle) and 4, 5, 6 down the column of the "T" (5 is the pentagon). An event is a pair (a, b) where the moth lands on a and moves to b. The possible outcomes are $(1, 2)$, $(2, 3)$, $(2, 4)$, $(2, 1)$, $(3, 2)$, $(4, 2)$, $(4, 5)$, $(5, 4)$, $(5, 6)$ and $(6, 5)$. The successful outcomes are $(4, 5)$ and $(6, 5)$ (they have to end up on 5). However, the outcomes have different individual probabilities. They are not equally likely events, therefore simple counting of events does not suffice to solve the problem. The probability for event $(4, 5)$ is $\frac{1}{6}$ (one chance in six of landing on 4) times $\frac{1}{2}$ (one chance in two of moving to 5), i.e., $\frac{1}{6}\frac{1}{2} = \frac{1}{12}$. The probability for event $(6, 5)$ is $\frac{1}{6}$ (one chance in six of landing on 4) times 1 (only one move is possible), i.e., $\frac{1}{6}(1) = \frac{1}{6}$. The total probability is the sum $\frac{1}{12} + \frac{1}{6} = \frac{1}{4}$.

32. **(Triangles)** The biggest rectangle in the figure consists entirely of right triangles. Find the perimeter of this rectangle in feet, given $\tan A = \frac{2}{5}$, $\tan B = \frac{4}{5}$, $\tan C = \tan D = \frac{1}{2}$, $\tan E = 4$, $\tan F = 1$, $\tan G = \frac{1}{3}$ and $x = 15$ ft.

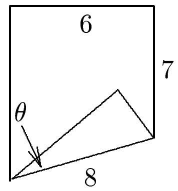


[**a**] 138 (b) 139 (c) 140 (d) 142 (e) 143

Def: *$\tan A$ in a right triangle is the quotient of opposite and adjacent sides.*

Solution to 32. The tangent of an angle is the quotient of the opposite and adjacent sides. Using this fact allows the sides to be determined as 33 and 36 feet, hence the perimeter is 138. A difficulty arises with angle F, because to find the legs of that triangle it is necessary to observe that the legs are equal ($\tan F = 1$). Let x be the unknown length of the equal legs. By this stage the horizontal dimension of the rectangle is known to be 36ft. So $\tan G = \frac{1}{3} = \frac{x}{36-x}$, which determines $x = 9$ and finally the vertical dimension of the rectangle is 33ft.

33. **(Folding)** A rectangular sheet of paper is folded across a corner as shown in the figure, making angle θ. Find $\tan\theta$.



(a) $\frac{6}{7}$ (b) $\frac{3}{4}$ (c) $\frac{7}{8}$ [**d**] $\frac{\sqrt{7}}{3}$ (e) $\frac{\sqrt{7}}{4}$

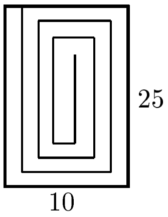
Solution to 33. When unfolded, the paper has a crease along the hypotenuse of a right triangle. The missing side x satisfies by the Pythagorean theorem $x^2 + 6^2 = 8^2$, which can be solved for $x = 2\sqrt{7}$. The tangent is the quotient of the sides of the right triangle, $\tan\theta = 2\sqrt{7}/6 = \sqrt{7}/3$.

34. **(Jackpot)** A jackpot of \$47 is divided into piles A, B, C, D so that A and B together have \$27, A and C have \$25 and finally A and D have \$23. Find the number of dollars in A.

(a) 13 [**b**] 14 (c) 15 (d) 16 (e) 17

Solution to 34. There are four equations in four unknowns, $A+B+C+D=47$, $A+B=27$, $A+C=25$, $A+D=23$. Add the last three to obtain $3A+B+C+D=75$. Subtract the first from it to get $2A=28$ and $A=14$.

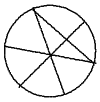
35. **(Paper helix)** A paper sheet 10×25 is cut by the method shown in the figure. By keeping the scissors always 1 unit from the edge, cutting results in a ribbon of 1-unit wide rectangles joined at right angles, which when lifted, forms a helix (a coil spring). Find the total perimeter.



(a) 216 (b) 432 (c) 430 [**d**] 502 (e) 503

Solution to 35. On the first path, the scissors travels 24. Then 8, 23, 7, 22 and so on, the total being $24+8+23+7+22+\cdots+1+16=216$. The scissors creates two edges simultaneously, therefore the created perimeter is $216+216=432$. The 10×25 rectangle perimeter is 70, making the total perimeter $432+70=502$.

36. **(Pizza)** Jeff cuts unequal pieces from a round pizza using 4 straight cuts. What is the greatest number of pieces he can cut?



(a) 8 (b) 9 (c) 10 [**d**] 11 (e) 12

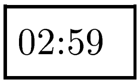
Solution to 36. By considering cases, 11 is the maximum. This is found by making each new cut cross the maximum number of previous cuts.

37. **(Chicken pen)** A triangular pen is constructed from 50 feet of fencing, using sides of length 11.5, 9.4 and x feet. The x actually used appears below. Find it.

(a) 30ft (b) 29.1ft (c) 22ft (d) 21ft [**e**] 16ft

Solution to 37. The two sides add to $11.5+9.4=20.9$. There is $50-20.9=29.1$ft of fencing remaining, but an allowed dimension must satisfy the triangle inequality. All given dimensions except 16 violate the triangle inequality.

38. **(Clock)** A 12-hour digital clock has 720 different displays, 60 per hour. How many displays have the sum of the digits greater than 15?



(a) 100 (b) 116 [**c**] 120 (d) 124 (e) 125

Solution to 38. The greatest sum of digits that can result from the hours digits is 9 and from the minutes digits $5+9=14$. The times 1:00 to 1:59 will therefore contribute no displays with sum of the digits greater than 15.

The times 2:00 to 3:00 produce only the display 2:59pm, giving just one qualifying time. The next hour produces times 3:49, 3:58 and 3:59. The following hour produces times 4:39, 4:48, 4:49, 4:57, 4:58 and 4:59. The pattern emerges as $0+1+3+6+10+15+21+27+33+0+1+3\cdots$, which is the sum of terms $n(n-1)/2$ from $n=1$ to $n=9$ less $1+3$ plus $0+1+3$. The total is 120 displays (not 124 as often claimed).

39. **(Time)** Find the *exact* time between 4 and 5 when the minute hand of a watch is precisely 5 minutes in advance of the hour hand.

(a) $4{:}26\frac{1}{2}$ (b) $4{:}26\frac{3}{4}$ (c) $4{:}27\frac{1}{11}$ [**d**] $4{:}27\frac{3}{11}$ (e) $4{:}28\frac{1}{11}$

Solution to 39. Let M be the reading of the minute hand in minutes after 4:00. Let H be the reading of the hour hand, read on the minutes scale. The time 4:00 corresponds to $H=20$. And H advances 5 when the hour hand moves from 4 to 5. Then at a time t minutes after 4:00, $0\le t\le 60$, we have $M=t$ and $H=20+t/12$. The time sought is defined by $M=H+5$. Inserting relations gives $t=25+t/12$ which is solved for $t=300/11=27+3/11$ minutes after 4.

An alternative solution, in terms of angles in degrees, can be constructed as follows. Let x be in minutes. The hour hand is at angle $120+30(x/60)$, the minute hand is at angle $360(x/60)$ and the clock position 5 minutes on the dial after 4:00pm is 30. The equation to determine x is $360(x/60)-120-30(x/60)=30$. Then $x=300/11$, giving the same answer as above.

40. **(Intercepts)** How many lines through $(4,3)$ have intercepts $(a,0)$ and $(0,b)$ with a and b being positive integers?

(a) 3 (b) 5 [**c**] 6 (d) 7 (e) Infinitely many

Solution to 40. Draw a line through $(a,0)$, $(4,3)$ and $(0,b)$. The slopes of the two segments in the figure are $3/(4-a)$ and $(b-3)/(-4)$. They are equal, which gives the equation $(a-4)(b-3)=12$. By geometry, the values of a are restrained to $5\le a$ and the values of b are restricted to $b\ge 4$. The question reduces to determining which positive integers p and q satisfy $pq=12$. This finite list is easily written down. Then $a=p+4$ and $b=q+3$. The possible pairs $[a,b]$ are $[5,15]$, $[6,9]$, $[7,7]$, $[8,6]$, $[10,5]$, $[16,4]$. There are six possible lines.

1995 Senior Exam Grade 10 with Key and Solutions

Answers to 1995 Exam Grade 10			
1. **(d)**	11. **(a)**	21. **(b)**	31. **(e)**
2. **(a)**	12. **(a)**	22. **(b)**	32. **(d)**
3. **(c)**	13. **(c)**	23. **(d)**	33. **(a)**
4. **(a)**	14. **(c)**	24. **(b)**	34. **(b)**
5. **(d)**	15. **(c)**	25. **(a)**	35. **(d)**
6. **(b)**	16. **(b)**	26. **(c)**	36. **(d)**
7. **(c)**	17. **(c)**	27. **(a)**	37. **(d)**
8. **(b)**	18. **(c)**	28. **(e)**	38. **(b)**
9. **(b)**	19. **(b)**	29. **(a)**	39. **(c)**
10. **(b)**	20. **(b)**	30. **(c)**	40. **(e)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Coolant)** Mix 3, 5 and 7 liters of coolant at temperatures 60°, 67° and 73° , respectively, to create a mixture at x degrees. Find x to the nearest degree.

(a) 65 (b) 66 (c) 67 [**d**] 68 (e) 70

Solution to 1. The mixture equation is $(3+5+7)x = 3(60)+5(67)+7(73)$. The answer is $x = 342/5 = 68.4$. The closest answer is 68.

2. **(Tangent)** Find the equation of the line through $(4,3)$ and tangent to the circle of radius 5 and center $(0,0)$.

[**a**] $4x+3y=25$ (b) $3x+4y=24$ (c) $4x-3y=7$ (d) $3x-4y=0$ (e) None

Solution to 2. The equation is $y-3=m(x-4)$. The point $(4,3)$ is on the circle, therefore m is the negative reciprocal of the slope of the radial line through $(4,3)$. That line has slope $3/4$, which implies $m=-4/3$.

3. **(Roots)** Apply the rational root theorem to the cubic $x^3-8x^2-7935=0$ and match a *candidate* for a root from the list below.

(a) 35 (b) 71 [**c**] 115 (d) 2643 (e) -343

Solution to 3. By the theorem, a potential root must divide 7935. The only divisor from the list is 115.

4. **(Guessing)** Jack guesses on 10 true-false questions. Find the probability that he gets exactly 8 correct.

[**a**] 45/1024 (b) 4/5 (c) 1/8 (d) 33/1024 (e) 45/512

Solution to 4. There are 2^{10} possible answer sets. The 10 questions can be marked with 8 correct and 2 incorrect in $1+2+\cdots+9 = 9\cdot 10/2$ ways. The probability is 45/1024.

5. **(Determinant)** Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ a & 1 & -2 \\ a^2 & 1 & 4 \end{vmatrix}$.

(a) $2+5a-a^2$ (b) $(3+a)(2+a)$ (c) $-(1+a)(2+a)$ [**d**] $(3-3a)(2+a)$ (e) None

Solution to 5. Since $a=1$ makes the determinant zero, because of equal columns, then only $(3-3a)(2+a)$ is a candidate. The determinant has value $(-2-1)(-2-a)(1-a)$ by direct expansion using Sarrus' rule or a cofactor expansion along column one.

6. **(Ratio)** The perimeter of a triangle is 72 units and the sides have measures in the ratios 3:4:5. Find the number of units in the shortest side.

(a) 12 [**b**] 18 (c) 24 (d) 30 (e) None

Solution to 6. The measures of the sides can be written as $3x$, $4x$ and $5x$. The perimeter is $3x+4x+5x=72$. Then $x=6$, but x is not a side; the sides are $3x=18$, $4x=24$ and $5x=30$.

7. **(Straws)** Five straws of lengths 3, 4, 5, 6 and 12 units are used to form different triangles. How many have perimeter divisible by 3?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) None

Solution to 7. There are 4 possible triangles, 3-4-5, 3-4-6, 3-5-6 and 4-5-6. Only two have perimeter divisible by 3.

8. **(Polygons)** Which exterior angle is impossible for a regular polygon?

(a) 9° [**b**] 16° (c) $22\frac{1}{2}^\circ$ (d) 45° (e) $51\frac{3}{7}^\circ$

Solution to 8. The number n of sides of a regular polygon and the number x of degrees in an exterior angle satisfy the relation $n=360/x$. All of the answers x produce an integer for n, except for $x=16$, which gives the impossibility $n=360/16=22.5$.

9. **(Pentagon)** Find x, given a pentagon has exterior angles 63, 75, 58, x, $3x$ (in degrees).

(a) 39° [**b**] 41° (c) 43° (d) 45° (e) None

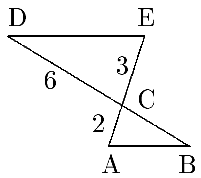
Solution to 9. The sum of the exterior angles must be 360°. This produces the equation $360 = 63+75+58+x+3x$ for the unknown x. Solving, $x=41$.

10. **(Midway)** The line $5x+8y=4.5$ is midway between and parallel to the parallel lines $5x+8y=3$ and $5x+8y=c$. Which is a possible value of c?

(a) 5 [**b**] 6 (c) 7 (d) 8 (e) 9

Solution to 10. Graph the lines. The "midway" requirement means $\frac{1}{2}(3+c)=4.5$, giving $c=6$.

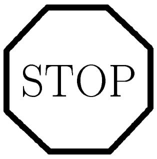
11. **(Geometry)** In the diagram, let segments $\overline{AB}$ and $\overline{DE}$ be parallel and let $AC=2$, $CD=6$ and $CE=3$. Find BC.



[**a**] 4 (b) 5 (c) 6 (d) 7 (e) None

Solution to 11. Let $x = BC$. The triangles ABC and EDC are similar, therefore $\frac{3}{2} = \frac{6}{x}$, which can be solved for $x = 4$.

12. **(Stop)** A stop sign with eight sides is to be cut from sheet metal that is stocked in square pieces. Given the stop sign edges are 1 ft, find the square footage of an uncut stock piece which minimizes waste.



[**a**] $3 + 2\sqrt{2}$ (b) $2 + 3\sqrt{2}$ (c) $2 + 2\sqrt{3}$
(d) $4 + 2\sqrt{5}$ (e) 8

Solution to 12. The minimum square will touch on four sides of the octagon. The stop sign is made by cutting off corners of the square, leaving a right triangle of hypotenuse 1 with legs x and x. The Pythagorean theorem implies $2x^2 = 1$ or $x = 1/\sqrt{2}$. The side of the square is $1 + 2x = 1 + \sqrt{2}$ and the area is $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$.

13. **(Population)** In 1990, the population of a city was 12298, a decrease of $8\frac{1}{3}\%$ from its 1980 population. The 1980 population was an increase of $7\frac{1}{2}\%$ from its 1970 population. Find the 1970 population.

(a) 12360 (b) 12380 [**c**] 12480 (d) 12500
(e) None

Solution to 13. Let P and Q be the populations in 1970 and 1980, respectively. Then $8\frac{1}{3}\%$ is $1/12$ and $7\frac{1}{2}\%$ is $3/40$, giving $12298 = Q - Q/12$ and $Q = P + 3P/40$. Then $P = \frac{40}{43}Q = \frac{40}{43}\frac{12}{11}12298 = 12480$.

14. **(Integers)** If the product of three positive consecutive integers is divided by each of them in turn, then the sum of the three quotients is 74. Find the smallest of the three integers.

(a) 2 (b) 3 [**c**] 4 (d) 5 (e) -6

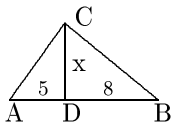
Solution to 14. Let $N = x(x + 1)(x + 2)$. Then $\frac{N}{x} + \frac{N}{x+1} + \frac{N}{x+2} = 74$. This gives the quadratic equation $x^2 + 2x - 24 = 0$ which has roots 4 and -6. The answer is $x = 4$, because $x > 0$.

15. **(Symmetry)** How many lines of symmetry are there for the region inside a non-rectangular rhombus?

(a) 4 (b) 3 [**c**] 2 (d) 1 (e) 0

Solution to 15. A line of symmetry is a line that can be drawn through the figure so that it can be folded along the line and the two halves match exactly. Only the diagonals of a rhombus have this property, unless the rhombus is a square.

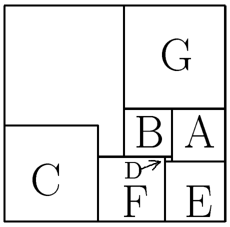
16. **(Altitude)** Given right triangle ABC with hypotenuse AB and measures $AD = 5$, $DB = 8$, then find altitude $x = CD$.



(a) $\sqrt{10}$ [**b**] $2\sqrt{10}$ (c) $2\sqrt{11}$ (d) $\frac{5}{3}\sqrt{13}$
(e) $\frac{7}{4}\sqrt{13}$

Solution to 16. The hypotenuse b of the small triangle can be computed by the geometric mean equation $\frac{5}{b} = \frac{b}{13}$. Then $b^2 = 65$. By the Pythagorean theorem, $b^2 = 5^2 + x^2$, hence $x^2 = 40$ and $x = 2\sqrt{10}$. This solution is equivalent to a trigonometric approach which equates the tangents of angles CAB and CBD.

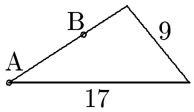
17. **(Area)** In the big square figure, assume shapes A through G are also squares. If A has area 64 cm^2 and B has area 49 cm^2, then find the area in cm^2 of square C.



(a) 100 (b) 121 [**c**] 169
(d) 225 (e) 324

Solution to 17. Squares A and B have sides 8 and 7, respectively. Square D has side 1. Then square E has side 9 and to its left square F has side 10. Square G has side $7+8 = 15$, due to squares A and B filling a side. Then the main square has side $15+8+9 = 32$. Finally, square C has side $32 - 10 - 9 = 13$ with area $13^2 = 169$.

18. **(Triangle)** The triangle in the figure has sides of 14, 9 and 17 units. Point B is on the inscribed circle. Find the number of units in AB.



(a) 9 (b) 10 [**c**] 11 (d) 12 (e) None

Solution to 18. Let $x = AB$. Let the sides of lengths 14, 9 and 17 be divided in ratios $x{:}z$, $z{:}y$ and $y{:}x$, respectively, by the contact points with the inscribed circle. We used: *If two segments from the same exterior point are tangent to a circle, then they are congruent.* Then $x + z = 14$, $y + z = 9$ and $y + x = 17$. Solving these three equations in three unknown gives $x = 11$, $y = 6$, $z = 3$.

19. **(Midpoint)** Given $A(x, -2x + 1)$ and $B(-x, 2x - 1)$, find the midpoint of segment $\overline{AB}$.

(a) (x, x) [**b**] $(0, 0)$ (c) $(-1, 0)$ (d) $(0, -1)$
(e) None

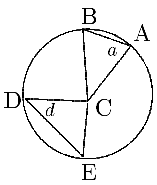
Solution to 19. The midpoint is $((x-x)/2, (-2x+1+2x-1)/2) = (0, 0)$.

20. **(Polygon)** The sum of the measures of the interior angles in a polygon is 24840. Find the number of sides.

(a) 138 [**b**] 140 (c) 142 (d) 144 (e) 145

Solution to 20. The equation for the number n of sides is $180(n - 2) = 24840$. Solving gives $n = 140$.

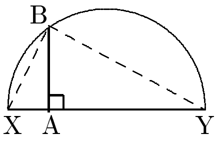
21. **(Angles)** Given angles $a = 50°$ and $d = 65°$ as in the figure, then find the sum of the measures of angles ECA and BCD.



(a) 220 [**b**] 230 (c) 232 (d) 245
(e) 250

Solution to 21. The two triangles are isosceles. The base angles adjacent to the circle are 65, 65 and 50, 50, respectively. So the central angles used by the triangles are $180-65-65 = 50$ and $180-50-50 = 80$. These two central angles sum to $50+80 = 130$. Then $a + d + 130 = 360$, or $a + d = 230$.

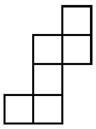
22. **(Chord)** If B is a point on a semicircle XBY with $\overline{AB} \perp \overline{XY}$ at point A, $XA = 2\frac{1}{12}$ and $AY = 12$, then find BY.



(a) $\frac{25}{2}$ [**b**] 13 (c) $9\sqrt{2}$ (d) $8\sqrt{3}$ (e) $6\sqrt{5}$

Solution to 22. Let x be the unknown length BY. Let $u = XB$ and $v = AB$. The Pythagorean theorem implies $x^2 = v^2 + 12^2$ and $u^2 = (2 + \frac{1}{12})^2 + v^2$. Since angle B is 90°, then the Pythagorean theorem implies $u^2 + x^2 = (12 + 2 + \frac{1}{12})^2$. Eliminating between the two equatons for x^2 gives $u^2 + v^2 = 50 + (25/12)^2$. Using the other equation for u^2 and v^2 gives $v = 5$. Then $x^2 = v^2 + 12^2 = 13^2$ and $x = 13$.

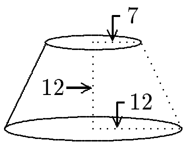
23. **(Perimeter)** The six congruent squares in the figure have total area 486 cm^2. Find the perimeter of the figure in cm.



(a) 120 (b) 122 (c) 124 [**d**] 126 (e) 130

Solution to 23. Let x be the side of a square. Then $6x^2 = 486$, giving $x = 9$. The perimeter, using the figure, is $14x = (14)(9) = 126$.

24. **(Frustrum)** The figure is a right circular cone of base radius 12 cm that has been cut off parallel to the base at height 12 cm. Find the lateral surface area in cm^2.



(a) 245π [**b**] 247π (c) 248π (d) 250π (e) 256π

Solution to 24. The portion shown in the figure is called a *frustrum of a cone*. The lateral area of a frustrum is the difference of the lateral areas of the larger cone and the smaller cone. Lateral areas are computed for cones by the formula $L = \pi R\ell$, where R is the radius of the base and ℓ is the slant height of the cone. The lateral area does not include the area of the circular base. The slant height of the frustrum is $13 = \sqrt{5^2 + 12^2}$, using the Pythagorean theorem. The slant height x of the smaller excised cone must satisfy by similar triangles the equation $\frac{x}{7} = \frac{13}{5}$. The difference of the lateral areas of the larger and smaller cones is $12\pi(13 + x) - 7\pi x = 156\pi + 5\pi x = 156\pi + 91\pi = 247\pi$.

25. **(Prism)** Find the surface area in cm^2 of a right prism of height 15 cm and right triangular base with legs of 3 cm and 4 cm.

[**a**] 192 (b) 180 (c) 240 (d) 190 (e) 220

Solution to 25. The surface area is the sum of the areas of the faces. There are two triangular faces, each of area $\frac{1}{2}(3)(4)$, making 12. There are three rectangular faces, of dimensions 15×3, 15×4 and $15 \times \sqrt{3^2 + 4^2}$. The last dimension is obtained by an application of the Pythagorean theorem. The total area is $12 + (15)(3 + 4 + 5) = 192$.

26. **(Complex)** Find the conjugate inverse of the complex number $\frac{1}{z^2}$ where $z = \frac{i}{1-i}$ and $i = \sqrt{-1}$.

(a) i (b) $-i$ [**c**] $\frac{i}{2}$ (d) $\frac{-i}{2}$ (e) None

Solution to 26. The inverse is z^2. The conjugate of z^2 is the square of the conjugate of z. Then $\overline{z} = \frac{-i}{1+i}$ implies $\overline{z}^2 = \frac{i^2}{(1+i)^2} = \frac{-1}{2i} = \frac{i}{2}$.

27. **(Magnitude)** Let $f(x) = \sin^2(x) + \cos(x)$, $g(x) = \frac{2x^2+100x+11}{(3x-1)(x+1)}$, $h(x) = e^{x^2}\cos(x)$. If x is an odd integer multiple of $\pi/2$ and $x > 10^{10}$, then which function value below is largest?

[**a**] $f(x)$ (b) $g(x)$ (c) $h(x)$ (d) $f(x)g(x)$ (e) $f(x)h(x)$

Solution to 27. All values of $h(x)$ are zero. The value of $g(x)$ is about 2/3, because by the division algorithm for polynomials, $g(x) = 2/3 + r(x)$ where $r(x)$ is nearly zero. The value of $f(x)$ is 1 for odd integer multiples of $\pi/2$. The largest value is given by $f(x)$.

28. **(Disk)** Select from the list below a point on the line $2x + 3y = 1$ which is also inside the circle of radius 3 and center $(-1, 2)$.

(a) $(1, -\frac{1}{3})$ (b) $(-4, 3)$ (c) $(\frac{9}{8}, -\frac{5}{12})$ (d) $(\frac{5}{4}, -\frac{1}{2})$ [**e**] $(-\frac{15}{4}, \frac{17}{6})$

Solution to 28. The circle equation is $(x + 1)^2 + (y - 2)^2 = 9$. Insert the line equation $y = (1 - 2x)/3$ into the circle equation to find the x-values where the line crosses the circle. They occur at approximately 0.9 and −3.9. So it is difficult to eliminate any of the answers. The best attack is to check which ones are on the line and then test them against the inequality $(x + 1)^2 + (y - 2)^2 < 9$. All five points are on the line $2x + 3y = 1$. The values of $(x + 1)^2 + (y - 2)^2$ at the five points are 9.4, 10, 10.4, 9.4, 8.3. Therefore, $(-\frac{15}{4}, \frac{17}{6})$ qualifies as on the line and inside the circle.

29. **(Series)** Sum the series $\frac{1}{2} + (\frac{1}{3} + \frac{2}{3}) + (\frac{1}{4} + \frac{2}{4} + \frac{3}{4}) + (\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}) + \cdots + (\frac{1}{100} + \frac{2}{100} + \frac{3}{100} + \cdots + \frac{99}{100})$.

[**a**] 2475 (b) 4950 (c) 2476 (d) 4952 (e) None

Solution to 29. The terms in parentheses can be simplified by using the formula $1 + 2 + 3 + \cdots + n = n(n+1)/2$ (Gauss' childhood sum formula). The simplifications are $\frac{1}{2}, \frac{2(3)/2}{3}, \frac{3(4)/2}{4}, \frac{4(5)/2}{5}, \ldots, \frac{99(100)/2}{100}$. This is half of $1+2+3+4+\cdots+99 = 99(99+1)/2 = 4950$. The series sum is 2475.

30. **(Polynomials)** Let p be a quintic polynomial in x with roots $1-2i$, $1+2i$, $\sqrt{2}$, $-\sqrt{2}$ and 1. Assume $p = 160$ when $x = -1$. Find p when $x = \frac{3}{2}$.

(a) $\frac{23}{2}$ (b) 12 [**c**] $\frac{25}{2}$ (d) 13 (e) $\frac{125}{4}$

Solution to 30. By the factor theorem, $(x-1+2i)$, $(x-1-2i)$, $(x-\sqrt{2})$, $(x+\sqrt{2})$ and $(x-1)$ are factors of p. The polynomial p must be a constant multiple of $(x^2+4)(x^2-2)(x-1)$. Briefly, this means $p = c(x^2+4)(x^2-2)(x-1)$ for some constant c. In expanded form, $p = c(x^5 - x^4 + 2x^3 - 2x^2 - 8x + 8)$. Since $p = 160$ at $x = -1$, then $160 = c(1+4)(1-2)(-1-1)$ or $c = 16$. The value of p at $x = \frac{3}{2}$ is $16(\frac{9}{4}+4)(\frac{9}{4}-2)(\frac{3}{2}-1) = \frac{25}{2}$.

See *infra* for problems 31-40.

1995 Senior Exam Grade 11 with Key and Solutions

Answers to 1995 Exam Grade 11			
1. **(e)**	11. **(b)**	21. **(e)**	31. **(e)**
2. **(a)**	12. **(a)**	22. **(e)**	32. **(d)**
3. **(a)**	13. **(c)**	23. **(d)**	33. **(a)**
4. **(b)**	14. **(d)**	24. **(e)**	34. **(b)**
5. **(a)**	15. **(e)**	25. **(b)**	35. **(d)**
6. **(d)**	16. **(d)**	26. **(c)**	36. **(d)**
7. **(b)**	17. **(a)**	27. **(d)**	37. **(d)**
8. **(e)**	18. **(d)**	28. **(d)**	38. **(b)**
9. **(a)**	19. **(e)**	29. **(e)**	39. **(c)**
10. **(a)**	20. **(c)**	30. **(a)**	40. **(e)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Distance)** Find the distance between the points $(-1, 2, 4)$ and $(1, -2, 2)$.

(a) $2\sqrt{2}$ (b) $2\sqrt{3}$ (c) 4 (d) $2\sqrt{5}$ [**e**] $2\sqrt{6}$

Solution to 1. Use the distance formula $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$ to obtain $\sqrt{((2)^2 + (-4)^2 + (-2)^2} = 2\sqrt{6}$.

2. **(Area)** Find the approximate area of a regular pentagon of perimeter 45 cm.

[**a**] $101.25\cot(\frac{\pi}{5})$ (b) $101.25\cot(\frac{2\pi}{5})$
(c) $101\cot(\frac{\pi}{5})$ (d) $101\cot(\frac{2\pi}{5})$ (e) $100\cot(\frac{\pi}{5})$

Solution to 2. The length of each side is $45/5 = 9$. The length of an apothem is $(9/2)/\tan(36°)$, because each central angle is $360/5 = 72$. The area is $\frac{1}{2}$ the perimeter times the apothem, or $\frac{45}{2}\frac{9}{2}\cot(36°)$. The radian measure of $36°$ is $\frac{\pi}{5}$.

3. **(Shadow)** When the angle of elevation of the sun is $44°$, the shadow cast by a building is 35 meters. Approximate the building height in meters.

[**a**] 33.8 (b) 35.1 (c) 36.1 (d) 36.5 (e) 36.9

Solution to 3. Draw a figure using a 44-46-90 triangle with legs 35 and x. Then $x/35 = \tan(44°)$. The answer is $x = 35\tan(44°) = 33.8$. If the angle was 45, then the answer would be $35\tan(45°) = 35(1) = 35$. The only choice is 33.8.

4. **(Planes)** How many planes are determined by five points in space, no four of which are coplanar?

(a) 8 [**b**] 10 (c) 11 (d) 14 (e) None

Solution to 4. Each three set of points determines a plane, but taking all combinations leads to duplicate planes. The distinct ones number 10 in all.

5. **(Traffic)** Three out of five cars arriving at a signal light have a green or yellow light. Estimate the probability that exactly 4 of the next 7 cars will have a red light.

[**a**] 0.19 (b) 0.20 (c) 0.22 (d) 0.24 (e) 0.29

Solution to 5. Let $p = \frac{2}{5}$, $q = \frac{3}{5}$. Then $p + q = 1$ and p is the probability of a red light for one car. The probability of exactly four out of the next seven being red is the term of the binomial expansion $(p+q)^7$ with power p^4q^3. This term is $\binom{7}{4} p^4q^3 = 35(\frac{2}{5})^4(\frac{3}{5})^3 = 0.193536$.

6. **(Rhombus)** A rhombus with each side 26 cm has one diagonal 48 cm. Find the number of centimeters in the other diagonal.

(a) 10 (b) 14 (c) 16 [**d**] 20 (e) 40

Solution to 6. The diagonals meet at right angles and bisect each other, therefore the other diagonal x must satisfy, by the Pythagorean theorem, $24^2 + (x/2)^2 = 26^2$. Solving, $x = 20$.

7. **(Rectangle)** Find the larger dimension of a rectangle of perimeter 32 units and area 48 square units.

(a) 11 [**b**] 12 (c) 13 (d) 14 (e) 4

Solution to 7. The sides a and b of the rectangle satisfy $2a+2b = 32$, which implies the dimensions are a and $16-a$. So the area is $a(16-a) = 48$. Trying numbers for a from 1 to 15 gives $a = 4$. The larger of 4 and $16-4$ is 12.

8. **(Volume)** Given a regular right prism 40 cm high whose cross section is a regular hexagon of perimeter 30 cm, find the volume in cm^3.

(a) $500\sqrt{3}$ (b) $1837\sqrt{2}$ (c) $612\sqrt{2}$ (d) 2598
[**e**] $1500\sqrt{3}$

Solution to 8. The volume is the base area B times the height 40. To find B, divide the hexagon into 6 equilateral triangles of side $30/6 = 5$ cm. Each equilateral triangle has area $25\sqrt{3}/4$, so six of them gives $B = 75\sqrt{3}/2$. The volume is $40B = (20)(75)\sqrt{3} = 2598.1$.

9. (Interest) Find the least number of full years for a deposit to double using continuous interest at 7% per annum. Use $\ln(2) = 0.693$.

[**a**] 10 (b) 9 (c) 8 (d) 7 (e) 6

Solution to 9. Let $x(t)$ be the amount and $x(0)$ the initial amount. Then $x(t) = x(0)e^{rt}$ where $r = 7/100$ is the per annum interest rate. Doubling means $x(t) = 2x(0)$, therefore the equation to be solved for t is $2x(0) = x(0)e^{rt}$. Cancel $x(0)$ and take logarithms to obtain $\ln(2) = \ln(e^{rt})$ and finally $t = \frac{1}{r}\ln(2) = \frac{100}{7}\ln(2)$, which is about 9.9 years.

10. (Distance) The curves $\frac{x^2}{9} + \frac{y^2}{16} = 1$ and $x + y = 1$ meet in exactly two points P and Q. Find the distance from P to Q.

[**a**] $\frac{96}{25}\sqrt{3}$ (b) $\frac{19}{5}\sqrt{3}$ (c) $\frac{48}{13}\sqrt{3}$ (d) $\frac{47}{13}\sqrt{3}$
(e) $4\sqrt{3}$

Solution to 10. The points P and Q have the form $(x_1, 1 - x_1)$ and $(x_2, 1 - x_2)$ where x_1 and x_2 are the roots of the quadratic equation $x^2/9 + (1-x)^2/16 = 1$. The distance is $\sqrt{(x_1 - x_2)^2 + (-x_2 + x_1)^2} = \sqrt{2}|x_1 - x_2|$. Since the roots are $(9 \pm 24\sqrt{6})/25$, the answer is $\sqrt{2}(48\sqrt{6}/25) = 96\sqrt{3}/25$.

11. (Inverse) Which expression below for $f(x)$ is also the expression for $f^{-1}(x)$?

(a) $\frac{1}{x^2}$ [**b**] $-x$ (c) $\log(x)$ (d) $\frac{x}{1+x}$ (e) None

Solution to 11. To test (a), let y be the given expression, swap x and y to get $x = 1/y^2$, then solve for y in terms of x. This results in $y = 1/\sqrt{x}$, therefore (a) does not qualify as the inverse of itself. Both (c) and (d) fail, but (b) works, because $y = -x$ swaps to $x = -y$ and then $y = -x$, reproducing the original equation.

12. (Sequence) Find the value of a_{12}/b_9, given $a_n = 32000(0.8)^{2n+1}$ and $b_n = 16000(0.4)^{3n-2}$.

[**a**] 2^{26} (b) 2^{25} (c) 2^{-26} (d) 2^{-25} (e) 2^{24}

Solution to 12. The values are $a_{12} = 32000(0.8)^{25}$ and $b_9 = 16000(0.4)^{25}$. Then $a_{12}/b_9 = 2(0.8)^{25}/(0.4)^{25} = 2(0.8/0.4)^{25} = 2^{26}$.

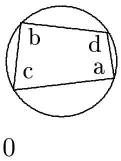
13. (Skid) Viktor measured skid marks when the brakes of his Volvo were applied at different speeds. Use a simple data fitting model to estimate how far in feet the Volvo would skid if the brakes were applied at 70 mph.

ft	mph
18	20
41	30
72	40
113	50
162	60

(a) 201 (b) 211 [**c**] 221 (d) 232 (e) 241

Solution to 13. The graph of the data looks quadratic. The best model seems to be $y = kx^2$ where y is the skid length in feet and x is the speed in mph. The first data point gives $k = 18/20^2 = 0.045$. Testing $y = 0.045x^2$ reproduces the table. The prediction is y at $x = 70$, which is $y = 0.045(70)^2 = 220.5$ feet.

14. (Inscribed) The quadrilateral inscribed in the circle has angles $a = 2x$, $b = 2x + 10$, $c = 2x - 10$, all in degrees. Find the measure of angle d in degrees.



(a) 85 (b) 95 (c) 100 [**d**] 105 (e) 110

Solution to 14. Opposite angles in an inscribed quadrilateral add to 180°. This produces from $a+b = 180$ the equation $2x + 2x + 10 = 180$ and finally $x = 85/2$. Then $c = 2x - 10 = 75$. Finally, $c + d = 180$, giving $d = 105$.

15. (Scaling) Given rectangle $ABCD$ where $A = (-2, 4)$, $B = (-2, -4)$, $C = (-6, -4)$, $D = (-6, 4)$, let T be the image of $ABCD$ obtained by applying the matrix product $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ to each point (x, y) of $ABCD$. Find the area of T in square units.

(a) 160 (b) 320 (c) 400 (d) 600 [**e**] 800

Solution to 15. The matrix applies a scale factor of 5 to the figure, therefore the edges of the image figure are 5 times longer than the original. Rectangle $ABCD$ has sides 4 and 8, so T is a rectangle with sides 20 and 40. The area of T is $(20)(40) = 800$.

16. (Bisector) Find the y-intercept of the perpendicular bisector of the segment joining $(9, 5)$ and $(-7, 11)$.

(a) 5 (b) $\frac{31}{6}$ (c) $\frac{11}{2}$ [**d**] $\frac{16}{3}$ (e) $\frac{17}{3}$

Solution to 16. The segment has slope $m = (11-5)/(-7-9) = -3/8$, therefore the line perpendicular to it has slope $-1/m = 8/3$. The midpoint of the segment is $(\frac{9+(-7)}{2}, \frac{5+11}{2}) = (1, 8)$. The perpendicular bisector therefore has equation $y - 8 = (8/3)(x - 1)$. It has y-intercept satisfying the equation $y - 8 = (8/3)(0 - 1)$, which gives $y = 16/3$.

17. (Systems) Count the number of solutions for

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 2 \\ 2 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

[**a**] Infinite (b) 0 (c) 1 (d) 2 (e) None

Solution to 17. The determinant is zero, therefore it has either 0 or infinitely many solutions. Since $x = 1$, $y = 0$, $z = 0$ works in the equation, there are infinitely many.

18. (Determinant) Given $c - a = 3x$ and $c - b = 2x$, then find in terms of x the value of

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

(a) $-3x^3$ (b) $3x^3$ (c) $-6x^3$ [**d**] $6x^3$ (e) None

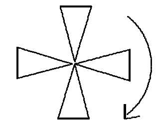
Solution to 18. The determinant has value $(c-a)(c-b)(b-a)$ by direct expansion using Sarrus' rule or a cofactor expansion along row one. The determinant is known in the literature as *Vandermonde's* *determinant*. The value of $b-a$ is obtained by subtracting $c-b$ from $c-a$. Substitution of $c-a=3x$, $c-b=2x$, $b-a=x$ gives the answer $6x^3$.

19. **(Geodesic)** Find the distance from the point $(2,3)$ to the line $3x+4y=7$.

(a) 2 (b) $\frac{12}{5}$ (c) $\frac{9}{4}$ (d) $\frac{55}{24}$ **[e]** $\frac{11}{5}$

Solution to 19. The distance from a point to a line is measured as the perpendicular distance, taken along a line normal to the given line and through the given point $(2,3)$. The slope of this line is the negative reciprocal of the slope of $3x+4y=7$. Using $y-y_0=m(x-x_0)$ as the line equation, then $x_0=2$, $y_0=3$, $m=-1/(-3/4)=4/3$ and the line is $y-3=(4/3)(x-2)$. The lines meet when $y=3+(4/3)(x-2)=(7-3x)/4$, which gives $x=17/25$ and $y=31/25$. The distance is $\sqrt{(2-17/25)^2+(3-31/25)^2}=11/5$.

20. **(Fan)** An electric fan is switched off, making n turns in the first second. Each additional second, it loses σ percent of its revolutions. Find the maximum number of fan revolutions.



(a) $102\frac{n}{\sigma}$ (b) $101\frac{n}{\sigma}$ **[c]** $100\frac{n}{\sigma}$ (d) $99\frac{n}{\sigma}$ (e) $98\frac{n}{\sigma}$

Solution to 20. The series whose sum is the maximum number of turns is

$$n\sum_{k=0}^{\infty}\left(1-\frac{\sigma}{100}\right)^k.$$

This is a geometric series of common ratio $r=1-\sigma/100$ with sum $n/(1-r)$. The sum simplifies to $100n/\sigma$. To derive the series formula, observe the number of revolutions during a later second is the number from the preceding second multiplied by the fraction $1-0.01\sigma$, because 0.01σ revolutions are lost.

21. **(Zeros)** Count the number of real roots x of the equation $\ln|x^2-3x+1|=0$.

(a) 0 (b) 1 (c) 2 (d) 3 **[e]** 4

Solution to 21. The equation is equivalent to two quadratic equations, $x^2-3x+1=1$ and $x^2-3x+1=-1$. The first has roots $x=0$ and $x=3$. The second has roots $x=1$ and $x=2$. There are four real roots.

22. **(Parallel)** The lines $5x+8y=9$ and $5x+8y=3$ are parallel. Find the distance between them.

(a) 5 (b) 6 (c) $\frac{3}{\sqrt{22}}$ (d) $\frac{\sqrt{2}}{\sqrt{5}}$ **[e]** $\frac{6}{\sqrt{89}}$

Solution to 22. A line perpendicular to both is $y-y_0=m(x-x_0)$ where (x_0,y_0) is on one of the lines and m is the negative reciprocal of the slope common to both lines. Choose $x_0=0$, $y_0=3/8$ (on $5x+8y=3$) and let $m=-1/(-5/8)=8/5$. Then $y-3/8=8x/5$ is the perpendicular line through $P=(0,3/8)$. Let Q be the intersection with the other line. Then the required distance is the distance from P to Q. Solving equations leads to $8x/5+3/8=y=9/8-5x/8$ with solution $x=30/89$, giving $Q=(30/89,651/712)$. Then $|P-Q|=\sqrt{(30/89)^2+(651/712-3/8)^2}=6/\sqrt{89}$.

23. **(Fractions)** Find D in the partial fraction expansion
$$\frac{p}{q}=\frac{A}{x+1}+\frac{B(x+1)+C}{x^2+2x+2}+\frac{D(x+1)+E}{(x^2+2x+2)^2},$$
if $p=x^2+3x+1$ and $q=(x+1)(x^2+2x+2)^2$.

(a) -1 (b) 0 (c) 1 **[d]** 2 (e) None

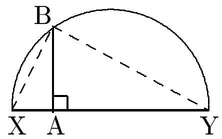
Solution to 23. The easiest method to apply is multiplication by $(x^2+2x+2)^2$ to clear the quadratic term. Then substitute a complex root of that factor, ie, $x=-1+i$. The result is a complex equation for the values D and E. Solving them gives $D=2$ and $E=1$. The other values are $A=-1$, $B=1$, $C=0$.

24. **(Disk)** Select from the list below a point on the line $2x+3y=1$ inside the circle of radius 3 and center $(-1,2)$.

(a) $(1,-\frac{1}{3})$ (b) $(-4,3)$ (c) $(\frac{9}{8},-\frac{5}{12})$ (d) $(\frac{5}{4},-\frac{1}{2})$ **[e]** $(-\frac{15}{4},\frac{17}{6})$

Solution to 24. The circle equation is $(x+1)^2+(y-2)^2=9$. Insert the line equation $y=(1-2x)/3$ into the circle equation to find the x-values where the line crosses the circle. They occur at approximately 0.9 and -3.9. So it is difficult to eliminate any of the answers. The best attack is to check which ones are on the line and then test them against the inequality $(x+1)^2+(y-2)^2<9$. All five points are on the line $2x+3y=1$. The values of $(x+1)^2+(y-2)^2$ at the five points are 9.4, 10, 10.4, 9.4, 8.3. Therefore, $(-\frac{15}{4},\frac{17}{6})$ qualifies as on the line and inside the circle.

25. **(Chord)** If B is a point on a semicircle XBY with $\overline{AB}\perp\overline{XY}$ at point A, $XA=2\frac{1}{12}$ and $AY=12$, then find BY.



(a) $\frac{25}{2}$ **[b]** 13 (c) $9\sqrt{2}$ (d) $8\sqrt{3}$ (e) $6\sqrt{5}$

Solution to 25. Let x be the unknown length BY. Let $u=XB$ and $v=AB$. The Pythagorean theorem implies $x^2=v^2+12^2$ and $u^2=(2+\frac{1}{12})^2+v^2$. Since angle B is 90°, then the Pythagorean theorem implies $u^2+x^2=(12+2+\frac{1}{12})^2$. Eliminating between the two equatons for x^2 gives $u^2+v^2=50+(25/12)^2$. Using the other equation for u^2 and v^2 gives $v=5$. Then $x^2=v^2+12^2=13^2$ and $x=13$.

26. **(Intercepts)** How many lines through $(4,3)$ have intercepts $(a,0)$ and $(0,b)$ with a and b being positive integers?

(a) 3 (b) 5 **[c]** 6 (d) 7 (e) Infinitely many

Solution to 26. Draw a line through $(a, 0)$, $(4, 3)$ and $(0, b)$. The slopes of the two segments in the figure are $3/(4-a)$ and $(b-3)/(-4)$. They are equal, which gives the equation $(a-4)(b-3) = 12$. By geometry, the values of a are restrained to $5 \le a$ and the values of b are restricted to $b \ge 4$. The question reduces to determining which positive integers p and q satisfy $pq = 12$. This finite list is easily written down. Then $a = p + 4$ and $b = q + 3$. The possible pairs $[a, b]$ are $[5, 15]$, $[6, 9]$, $[7, 7]$, $[8, 6]$, $[10, 5]$, $[16, 4]$. There are six possible lines.

27. **(Profit)** A plant produces x calculators per week at total cost $\frac{1}{5}x^2 + 15x + 200$ dollars. Given unit price $\frac{1}{3}(125 - x)$ dollars, find x to maximize revenue.

(a) 22 (b) 23 (c) 24 **[d]** 25 (e) 26

Solution to 27. The profit P is the revenue minus the cost. It is defined on $0 < x < \infty$ by $P = (x)(\frac{1}{3}(125 - x)) - (\frac{1}{5}x^2 + 15x + 200) = \frac{8}{15}(250 - (x - 25)^2)$. The maximum profit occurs at the vertex of this parabolic curve, which is at $x = 25$. The problem can also be worked by calculus, observing that P is a parabola with maximum when $P'(x) = 0$.

28. **(Functions)** Let $f(x) = \sqrt[6]{10}(10^x)$. Approximate $f(p)$ for $p = \dfrac{5u^2 + 2u + 3}{11u^2 - u + 2}$, $u = \dfrac{w}{1 + w}$ and $w = f(10^{12})$.

(a) 0.83 (b) 1 (c) 6.8 **[d]** 10 (e) Infinity

Solution to 28. Since w is about ∞, then $u = 1$ and $p = \frac{5+2+3}{11-1+2} = \frac{5}{6}$, approximately. Then $f(p) = 10^{1/6}10^{5/6} = 10$.

29. **(Series)** Find $\ln(\frac{x}{x-1})$ where x is the sum of the infinite series $\sum_{k=1}^{\infty} e^{-3n+3}$.

(a) e^2 (b) e^3 (c) 1 (d) 2 **[e]** 3

Solution to 29. The series has sum $1/(1 - r)$ where the common ratio is $r = e^{-3}$. Then the sum is $x = 1/(1 - e^{-3})$ and $x/(x - 1) = e^3$, making $\ln(\frac{x}{x-1}) = \ln(e^3) = 3$.

30. **(Jars)** Twelve identical straws are distributed at random into three jars. Find the approximate probability that the first jar contains three straws.

[a] $18(\frac{2}{3})^{11}$ (b) $19(\frac{2}{3})^{11}$ (c) $20(\frac{2}{3})^{11}$ (d) $21(\frac{2}{3})^{11}$ (e) $22(\frac{2}{3})^{11}$

Solution to 30. The first solution is by counting favorable events. The sample space contains 3^{12} events. If a favorable event occurs, then the remaining 9 straws are distributed among 2 jars. The number of events which forms the jar with three straws is $12!/(3!9!) = (12)(11)(10)/6 = (20)(11)$. The other two jars can be filled in 2^9 ways. So the favorable events amount to $(20)(11)(2^9) = (55)(2^{11})$. The probability is $(55)2^{11}/3^{12}$. The fraction 55/3 is 18.33, therefore the closest answer is $18(\frac{2}{3})^{11}$.

A simpler solution identifies the answer as arising from probabilities $p = \frac{1}{3}$ and $q = \frac{2}{3}$ and the binomial sum for $(p + q)^{12}$. The term that represents the desired probability is the one for p^3q^9, which is $\binom{12}{3} p^3q^9 = \frac{55}{3}(\frac{2}{3})^{11}$.

See *infra* for problems 31-40.

1995 Senior Exam Grade 12 with Key and Solutions

Answers to 1995 Exam Grade 12			
1. **(c)**	11. **(a)**	21. **(e)**	31. **(e)**
2. **(a)**	12. **(d)**	22. **(b)**	32. **(d)**
3. **(c)**	13. **(c)**	23. **(d)**	33. **(a)**
4. **(e)**	14. **(e)**	24. **(b)**	34. **(b)**
5. **(a)**	15. **(a)**	25. **(b)**	35. **(d)**
6. **(d)**	16. **(c)**	26. **(a)**	36. **(d)**
7. **(e)**	17. **(a)**	27. **(a)**	37. **(d)**
8. **(e)**	18. **(b)**	28. **(b)**	38. **(b)**
9. **(b)**	19. **(b)**	29. **(c)**	39. **(c)**
10. **(d)**	20. **(b)**	30. **(a)**	40. **(e)**

Instructions: Only one answer is correct. Scoring: 5(# right)+1(# blank)+0(# wrong). Time: 120 minutes.

1. **(Rain)** The Channel 7 weather program forecasts rain with probability of 60 percent. Find the odds it will not rain.

(a) $\frac{3}{5}$ (b) $\frac{2}{5}$ **[c]** $\frac{2}{3}$ (d) $\frac{3}{2}$ (e) $\frac{1}{5}$

Solution to 1. The 60% translates to $\frac{3}{5}$. Then *not rain* means the fraction is $\frac{2}{5}$. The *odds* equal the quotient of these two numbers, $\frac{2}{5}$ divided by $\frac{3}{5}$, or $\frac{2}{3}$.

2. **(Series)** Sum the infinite series
$$0.2 + 0.02 + 0.002 + 0.0002 + 0.00002 + \cdots.$$

[a] $\frac{2}{9}$ (b) $\frac{1}{3}$ (c) $\frac{5}{9}$ (d) $\frac{4}{9}$ (e) None

Solution to 2. The series is geometric with common ratio r. Write it as $0.2(1 + r + r^2 + \cdots)$ where $r = 0.02/0.2 = 0.1$. Then the sum is $0.2/(1 - r) = 0.2(10/9) = 2/9$.

3. **(Exponential)** Solve $12^{x-4} = 3^{x-2}$ for x, in terms of the constants $a = \log(3)$ and $b = \log(4)$.

(a) $2 + 2\frac{a}{b}$ (b) $3 + 2\frac{a}{b}$ **[c]** $4 + 2\frac{a}{b}$ (d) $12 - 2\frac{a}{b}$ (e) None

Solution to 3. The equation can be arranged as $3^{x-2}4^{x-4}3^{-2} = 3^{x-2}$, leaving after cancellation $4^{x-4} = 3^2$. Apply the logarithm to both sides to get $(x - 4)\log(4) = 2\log(3)$, which is $(x - 4)b = 2a$. Then $x = 4 + 2\frac{a}{b}$. The approximate answer is $x = 5.584962$.

4. (Division) The polynomial $x^5+12x^4+43x^3+36x^2-44x-48$ is perfectly divisible by x^2-1. Find the value of the quotient at $x=1$.

(a) 100 (b) 102 (c) 103 (d) 104 [e] 105

Solution to 4. The simplest solution is to do long division, obtaining the quotient $x^3+12x^2+44x+48$, whose value at $x=1$ is 105. A calculus solution, more complicated, is to form the limit at $x=1$ of the quotient of the quintic p and the quadratic q. The answer has to be $p'(1)=210$ divided by the value of $x+1$ at $x=1$. While this follows from the definition of derivative, it can be mechanically obtained from L'Hospital's rule applied to p/q, which is a "0/0" case.

5. (Roots) Given k and $2k$ are roots of the cubic equation $x^3+4x^2+9kx=90$, find the other root.

[a] 5 (b) 7 (c) -3 (d) -6 (e) -11

Solution to 5. The information reduces to $k^3+4k^2+9k^2=90$ and $8k^3+16k^2+18k^2=90$, which gives the equation $k^3+13k^2=90=8k^3+34k^2$. Solving for k gives the extraneous root $k=0$ (not a root of the cubic) and the useful one $k=-3$. So two roots are -3 and -6. Divide $x^3+4x^2-27x-90$ by $(x+3)(x+6)$ to get $x-5$. The last root is $x=5$.

6. (Sequence) Find the value of a_{12}/b_9, given $a_n=16000(0.8)^{2n+1}$ and $b_n=32000(0.4)^{3n-2}$.

(a) 2^{26} (b) 2^{25} (c) 2^{-25} [d] 2^{24} (e) 2^{-24}

Solution to 6. The values are $a_{12}=16000(0.8)^{25}$ and $b_9=32000(0.4)^{25}$. Then $a_{12}/b_9=\frac{1}{2}(0.8)^{25}/(0.4)^{25}=\frac{1}{2}(0.8/0.4)^{25}=(2^{24})$.

7. (Traffic) Four out of five cars arriving at a signal light have a green or yellow light. Estimate the probability that exactly 3 of the next 7 cars will have a red light.

(a) 0.03 (b) 0.05 (c) 0.07 (d) 0.09 [e] 0.11

Solution to 7. Let $p=\frac{1}{5}$, $q=\frac{4}{5}$. Then $p+q=1$ and p is the probability of a red light for one car. The probability of exactly three out of the next seven being red is the term of the binomial expansion $(p+q)^7$ with power p^3q^4. This term is $\begin{pmatrix} 7 \\ 3 \end{pmatrix} p^3q^4=35(\frac{1}{5})^3(\frac{4}{5})^4=0.11469$.

8. (Distance) The lines $5x+8y=9$ and $5x+8y=4$ are parallel. Find the distance between them.

(a) 5 (b) $\frac{13}{2}$ (c) $\frac{5}{9}$ (d) $\frac{\sqrt{10}}{6}$ [e] $\frac{5}{\sqrt{89}}$

Solution to 8. A line perpendicular to both is $y-y_0=m(x-x_0)$ where (x_0,y_0) is on one of the lines and m is the negative reciprocal of the slope common to both lines. Choose $x_0=0$, $y_0=1/2$ (on $5x+8y=4$) and let $m=-1/(-5/8)=8/5$. Then $y-1/2=8x/5$ is the perpendicular line through $P=(0,1/2)$. Let Q be the intersection with the other line. Then the required distance is the distance from P to Q. Solving equations leads to $8x/5+1/2=y=9/8-5x/8$ with solution $x=25/89$, giving $Q=(25/89,169/178)$. Then $|P-Q|=\sqrt{(25/89)^2+(169/178-1/2)^2}=5/\sqrt{89}$.

9. (Complex) Find the conjugate inverse of the complex number $\frac{1-z}{5+5z}$ where $z=\frac{i}{1-i}$ and $i=\sqrt{-1}$.

(a) $1+2i$ [b] $1-2i$ (c) $2-i$ (d) $2+i$ (e) None

Solution to 9. The inverse is $\frac{5+5z}{1-z}$. Then $\frac{5+5z}{1-z}=1+2i$. The conjugate is $1-2i$.

10. (Inverse) Find the determinant of $A^{-2}+A^{-1}$, given

$$A=\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & -x & -1 \end{bmatrix}$$

(a) 3 (b) $\frac{4}{9}$ (c) $9x$ [d] $\frac{x}{9}$ (e) $\frac{1}{9x}$

Solution to 10. The determinant rule $\det(AB)=\det(A)\det(B)$ can be applied to $A^{-2}+A^{-1}=A^{-2}(I+A)$ to give the answer as the product of $\det(A^{-2})$ and $\det(I+A)$. The first factor is $1/(\det(A))^2=1/9$ because of $\det(A)=3$ and $\det(I)=\det(A^2A^{-2})=\det(A)\det(A)\det(A^{-2})$. The second factor is a direct evaluation, which has value x, because $I+A$ adds 1's down the diagonal of A. The product of the factors is $x/9$.

11. (Systems) Count the number of solutions for

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 2 \\ 3 & -6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

[a] Infinite (b) 0 (c) 1 (d) 2 (e) None

Solution to 11. The determinant is zero, therefore it has either 0 or infinitely many solutions. Since $x=1$, $y=0$, $z=0$ works in the equation, there are infinitely many.

12. (Profit) A plant produces x calculators per week at total cost $\frac{1}{5}x^2+15x+200$ dollars. Given unit price $\frac{1}{3}(125-x)$ dollars, find x to maximize revenue.

(a) 22 (b) 23 (c) 24 [d] 25 (e) 26

Solution to 12. The profit P is the revenue minus the cost. It is defined on $0<x<\infty$ by $P=(x)(\frac{1}{3}(125-x))-(\frac{1}{5}x^2+15x+200)=\frac{8}{15}(250-(x-25)^2)$. The maximum profit occurs at the vertex of this parabolic curve, which is at $x=25$. The problem can also be worked by calculus, observing that P is a parabola with maximum when $P'(x)=0$.

13. (Logarithm) Count the number of possible solutions x of the equation $\log_4 x+\log_{x^4} 2=1$.

(a) 0 (b) 1 [c] 2 (d) 3 (e) infinite

Solution to 13. The left side is reduced by the formula $\log_b w=\log_2 w/\log_2 b$ into $\log_2 x/\log_2 4+\log_2 2/\log_2 x^4$ which is $\frac{u}{2}+\frac{1}{4u}$ where $u=\log_2 x$. Then the equation is equivalent to $\frac{u}{2} + \frac{1}{4u} = 1$, which can be solved for $u = 1 + 1/\sqrt{2}$ and $u = 1 - 1/\sqrt{2}$. There are two answers for x.

14. **(Disk)** Select from the list below a point on the line $2x + 3y = 1$ which is also inside the circle of radius 3 and center $(-1, 2)$.

(a) $(1, -\frac{1}{3})$ (b) $(-4, 3)$ (c) $(\frac{9}{8}, -\frac{5}{12})$ (d) $(\frac{5}{4}, -\frac{1}{2})$ [**e**] $(-\frac{15}{4}, \frac{17}{6})$

Solution to 14. The circle equation is $(x+1)^2 + (y-2)^2 = 9$. Insert the line equation $y = (1-2x)/3$ into the circle equation to find the x-values where the line crosses the circle. They occur at approximately 0.9 and -3.9. So it is difficult to eliminate any of the answers. The best attack is to check which ones are on the line and then test them against the inequality $(x+1)^2 + (y-2)^2 < 9$. All five points are on the line $2x + 3y = 1$. The values of $(x+1)^2 + (y-2)^2$ at the five points are 9.4, 10, 10.4, 9.4, 8.3. Therefore, $(-\frac{15}{4}, \frac{17}{6})$ qualifies as on the line and inside the circle.

15. **(Fractions)** Find E in the partial fraction expansion

$$\frac{p}{q} = \frac{A}{x+1} + \frac{B(x+1)+C}{x^2+2x+2} + \frac{D(x+1)+E}{(x^2+2x+2)^2},$$

if $p = x^2 + x + 1$ and $q = (x+1)(x^2+2x+2)^2$.

[**a**] -1 (b) 0 (c) 1 (d) 2 (e) None

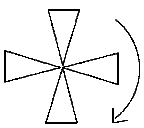
Solution to 15. Multiply by $(x^2+2x+2)^2$ to clear the quadratic term. Then substitute a complex root of that factor, ie, $x = -1 + i$. The result is a complex equation for the values D and E. Solving them gives $D = 0$ and $E = -1$. This solves the problem.

In a similar fashion, multiplying by $x + 1$ and then setting $x + 1 = 0$ results in $A = 1$. Then

$$\frac{x^2+2x+1}{(x+1)(x^2+2x+2)^2} + \frac{1}{(x^2+2x+2)^2} - \frac{1}{x+1} = \frac{B(x+1)+C}{x^2+2x+2}.$$

Substitute values $x = 0$ and $x = 1$ into this equation to get two equations in the two unknowns B and C. The equations are $B + C = -1$ and $2B + C = -2$. Solving gives $B = -1$ and $C = 0$.

16. **(Fan)** An electric fan is switched off, making n turns in the first second. Each additional second, it loses σ percent of its revolutions. Find the maximum number of fan revolutions.



(a) $102\frac{n}{\sigma}$ (b) $101\frac{n}{\sigma}$ [**c**] $100\frac{n}{\sigma}$ (d) $99\frac{n}{\sigma}$ (e) $98\frac{n}{\sigma}$

Solution to 16. The series whose sum is the maximum number of turns is

$$n \sum_{k=0}^{\infty} \left(1 - \frac{\sigma}{100}\right)^k.$$

This is a geometric series of common ratio $r = 1 - \sigma/100$ with sum $n/(1-r)$. The sum simplifies to $100n/\sigma$. To derive the series formula, observe the number of revolutions during a later second is the number from the preceding second multiplied by the fraction $1 - 0.01\sigma$, because 0.01σ revolutions are lost.

17. **(Iteration)** Let $f(z) = 0.5z + 2i$. Define $y_0 = f(1-i)$, $y_1 = f(y_0)$, $y_2 = f(y_1)$, ... Approximate the imaginary part of y_{30}.

[**a**] 4 (b) 5 (c) 8 (d) 10 (e) 16

Solution to 17. The iterates are $\frac{1}{4} + \frac{11}{4}i$, $\frac{1}{8} + \frac{27}{8}i$, $\frac{1}{16} + \frac{59}{16}i$, $\frac{1}{32} + \frac{123}{32}i$, $\frac{1}{64} + \frac{251}{64}i$, ... A more careful analysis shows that the imaginary part is $\frac{3}{2^n}$ plus the series $2\sum_{k=0}^{n-2}(1/2)^k$. The limit is 4. The right guess is 4 for y_{30}, because $\frac{3}{2^{30}}$ is about zero and $2\sum_{k=0}^{28}(1/2)^k = 4 - 4(1/2)^{29}$, which is very close to 4.

A more sophisticated attack on the problem uses iteration systems ideas and the theory of fixed points. Since 0.5 multiplies z, the function $f(z) = 0.5z + 2i$ is a contraction and it has a fixed point $z = f(z)$. Solving $z = 0.5z + 2i$ gives $z = 4i$, which is the limit of the iterates. This does not say what y_{30} equals, but given the choices in the problem, 4 is the only logical answer. See Merrill Precalculus (1994), page 730.

18. **(Magnitude)** Let $f(x) = \sin(x)$, $h(x) = \frac{1}{x}\ln(1+x)$, $g(x) = \dfrac{4x^2+100x+11}{(3x-1)(x+1)}$. For $x > 10^{10}$, which function value below is largest?

(a) $f(x^2)$ [**b**] $g(\sqrt{x})$ (c) $h(x)$ (d) $g(1/x)$ (e) $h(x^3)$

Solution to 18. The value of $f(x)$ is not larger than 1. The value of $g(x)$ is about 4/3, because by the division algorithm for polynomials, $g(x) = 4/3 + r(x)$ where $r(x)$ is nearly zero. The value of $g(1/x)$ is negative, about -11. The value of $h(x^3)$ is about 0. The largest value is given by $g(\sqrt{x})$, which is about 4/3.

19. **(Ellipse)** Find below one of the vertices of the ellipse $4x^2 + 25y^2 + 250y + 525 = 0$.

(a) $(-5, 0)$ [**b**] $(-5, -5)$ (c) $(0, -5)$ (d) $(0, 3)$ (e) $(0, 7)$

Solution to 19. Complete the square to write the equation in the equivalent form $4x^2 + 25(y^2 + 10y + 25) - 25^2 + 525$ and finally as $4x^2 + 25(y+5)^2 = 100$. The center is $(0, -5)$. The vertices are found by setting $x = 0$ or $y = -5$, which are the center coordinates, to obtain $(0, -7)$, $(0, -3)$, $(5, -5)$, $(-5, -5)$. The center can be found directly with calculus, eliminating the square completion step. Take the partial derivative of the equation in each variable and find the critical point: it will be the center. For example, the partial in the y variable is $50y + 250 = 0$ which gives $y = -5$.

20. **(Comet)** Halley's comet has an elliptical path. Approximate the length of the semi-minor axis in astronomical units (1 AU = 9.3×10^7 miles), given the eccentricity is 0.9672759 and the semi-major axis is 17.94104 AU.

(a) 4.9 [**b**] 4.6 (c) 4.2 (d) 4.0 (e) 3.8

Solution to 20. The key knowledge required is the formula $b^2 = a^2(1 - e^2)$. From this formula it is clear that b is the unknown and e, a are given in the problem. Then $b = 17.94104\sqrt{1 - (0.9672759)^2} = 4.552124232$. Calculator use on this problem is not required, because the square root can be estimated as 0.25, leaving only 4.6 as a reasonable answer.

21. **(Radius)** Find the radius of the circle that goes through the three points of intersection with the ellipse $x^2 + 4y^2 = 36$ and the parabola $x^2 + y = 3$.

(a) $\frac{223}{72}$ (b) $\frac{11}{4}$ (c) $\frac{3}{8}$ (d) $\frac{23}{6}$ [**e**] $\frac{27}{8}$

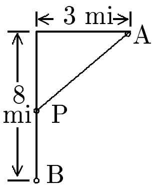
Solution to 21. Solve the parabola equation for $x^2 = 3 - y$, stuff it into the ellipse equation to get $3 - y + 4y^2 = 36$, which in normal form is the quadratic $4y^2 - y - 33 = 0$ with roots $y = 3$ and $y = -11/4$. Using the parabola equation, then $P = (0, 3)$, $Q = (\sqrt{23}/2, -11/4)$ and $R = (-\sqrt{23}/2, -11/4)$ are the three points of intersection. By geometry, the center of the circle through P, Q and R is on the line $x = 0$ at some point $y = c$. The distance formula can be used to get a quadratic equation for c, whose only solution is $c = -3/8$. The center of the circle is therefore at $(0, -3/8)$. The radius is $|3 - (-3/8)| = \frac{27}{8}$.

22. **(Train)** Commuter service costs \$1.50 per ride and 600 people ride the train daily. Each 5 cent fare increase causes 40 people to quit riding and each 5 cent fare decrease causes 40 more to ride. Approximate the fare which maximizes the revenue.

(a) \$1.11 [**b**] \$1.13 (c) \$1.15 (d) \$1.17 (e) \$1.19

Solution to 22. Let $P = 600 + N$ be the number of passengers, $F = 150 - 5N/40$ the fare per passenger in cents, $R = FP = (1200 - N)(600 + N)/8$ the total revenue. Using calculus, $0 = 8R' = 600 - 2N$ gives critical point $N = 300$ at which level the fare is $F = (1200 - 300)/8 = 225/2 = 112.5$. Without calculus, complete the square on the quadratic $(1200 - N)(600 + N)$ to write it as $810000 - (N - 300)^2$, which says the maximum is at the vertex $N = 300$.

23. **(Telephone)** A phone line runs from A to P to B, in the figure, with P selected to minimize the cost. Line $\overline{AP}$ and line $\overline{PB}$ have costs in the ratio 8:3. Approximate the cost-minimizing distance BP in miles.



(a) 6.2 (b) 6.4 (c) 6.6 [**d**] 6.8 (e) 7.0

Solution to 23. The cost of a stretch of line is the product of its length and the cost per unit length. Let $c > 0$ be chosen so that the expensive line costs $8c$ and the cheaper line costs $3c$, per unit length. Then the cost of the line is $y = 8c\sqrt{(x-8)^2 + 9} + 3cx$ where $x = BP$, $0 \le x \le 8$. The critical points satisfy $y' = 0$, which reduces to $55(8 - x)^2 = 81$ and finally $x = 8 - 9/\sqrt{55}$ (x must be positive). Checking endpoints, $y(8) = 48$, $y(0) = 8\sqrt{73}$. At the critical point, $y(8 - 9/\sqrt{55}) = 3\sqrt{55} + 24$, which is the minimum value of y on $0 \le x \le 8$. Then $BP = 8 - 9/\sqrt{55} = 6.8$.

This problem can be solved without calculus using a graphing calculator to localize the minimum. The plot should use $y = 8\sqrt{(8-x)^2 + 9} + 3x$ on $0 \le x \le 8$. A replot on $6.5 \le x \le 7.0$ will close in on the minimum. The minimum value of 46.25 at $x = 6.786$ is relatively simple to isolate.

24. **(Normal)** A function $f(x)$ has slope $-1/5$ at $(-1, 2)$. Find the x-intercept of the normal line at $(-1, 2)$.

(a) 7 [**b**] $-\frac{7}{5}$ (c) $\frac{9}{5}$ (d) $\frac{5}{2}$ (e) $-\frac{2}{5}$

Solution to 24. The tangent line is given by $y = 2 - (x + 1)/5$. The normal line is given by $y = 2 + 5(x + 1)$, because the slope of the normal line is the negative reciprocal of the slope of the tangent line. The x-intercept is the solution of the equation $0 = 2 + 5(x + 1)$. Then $x = -7/5$.

25. **(Inflection)** Determine a value of x for which the cubic $y - 1 = 2x^3 - x^2$ has an inflection point.

(a) $\frac{1}{3}$ [**b**] $\frac{1}{6}$ (c) 0 (d) $\frac{1}{2}$ (e) 1

Solution to 25. Calculus method: the critical points are roots of the quadratic $6x^2 - 2x$ ($x = 0$ and $x = 1/3$ are critical points). The inflection points can be found from the critical points of this quadratic, ie, from the equation $12x - 2 = 0$.

Graphical method, without calculus: the graph suggests a relative minimum near 0.3, a relative maximum near 0. A logical guess is that an inflection occurs at 1/6.

26. **(Derivative)** Find dy/dx at $x = 2$, given $y = \sqrt{z}$, $z = u/v$ and $u = 2x + 1$ and $v = x^2 - 3$.

[**a**] $\frac{-9}{\sqrt{5}}$ (b) $\frac{9}{\sqrt{5}}$ (c) $\frac{-18}{\sqrt{5}}$ (d) $\frac{18}{\sqrt{5}}$ (e) None

Solution to 26. The chain rule implies $y' = -\left(x^2 + 3 + x\right)\left(\frac{1+2x}{x^2-3}\right)^{-1/2}\left(x^2 - 3\right)^{-2}$. Substitution of $x = 2$ gives $\frac{-9}{\sqrt{5}}$.

27. **(Quartic)** Select from below the equation of a tangent line to an inflection point of the graph of the quartic $y = x^4 - 6x^3 + 12x^2 - 3x + 1$.

[**a**] $y = 5x + 1$ (b) $y = 5x - 5$ (c) $y = 7x + 2$ (d) $y = 7x - 9$ (e) $y = 5x + 13$

Solution to 27. The slope of the tangent line at x is $4x^3 - 18x^2 + 24x - 3$. Inflection points are those values of x such that $12x^2 - 36x + 24$ changes sign at x. The latter is $12(x^2 - 3x + 2) = 12(x - 2)(x - 1)$, which changes sign at $x = 1$ and $x = 2$. Points on the curve corresponding to these two values of x are $(1, 5)$, $(2, 11)$. The slopes there are 7 and 5, respectively. The tangent line equations are $y - 5 = 7(x - 1)$ and $y - 11 = 5(x - 2)$. Of these, only the second appears in the list, $y = 5x + 1$.

28. **(Integral)** Let f be the unique quadratic polynomial which passes through $(0, 3)$, $(1, -4)$ and $(2, -15)$. Find $\int_0^2 f(x)dx$.

(a) -9 [**b**] $-\frac{28}{3}$ (c) $-\frac{29}{3}$ (d) -10 (e) $-\frac{31}{3}$

Solution to 28. The exact value is given by Simpson's rule, according to numerical integration theory. The answer can be reported as the interval length 2 divided by 6 times the sum $(3)+4(-4)+(-15) = -28$, giving $-28(2)/6 = -28/3$.

Alternatively, compute the polynomial $f(x) = (x + 3)(1-2x) = -2x^2 - 5x + 3$ from the given information and integrate to obtain $-28/3$. To find the polynomial, let $f(x) = a + bx + cx(x - 1)$ and set up three equations in three unknowns a, b, c using the given data points: $3 = a$, $-4 = a + b$, $-15 = a + 2b + 2c$. Solve for a, b, c by elimination or Cramer's rule.

29. **(Integral)** Let f be nonnegative and have two continuous nonnegative derivatives. Assume $f(0) = 3$, $f(1) = 3.5$, $f(2) = 4$ and $f(3) = 5$. Select the smallest value y from below with $\int_0^3 f(x)dx \le y$.

(a) 10.1 (b) 10.4 [**c**] 11.5 (d) 12.5 (e) 13.0

Solution to 29. The idea is to approximate the integral from below using the Rectangular rule and from above using the Trapezoidal rule. The Rectangular rule implies $\int_0^3 f(x)dx \ge 10.5$. The approximation should use integrand values 3, 3.5 and 4. The assumption $f'(x) \ge 0$ implies f is increasing, therefore the rectangular approximation is from below. The Trapezoidal approximation uses three trapezoids, each of area $\frac{1}{2}(b + B)$ (b and B are the bases), to give upper estimate $\frac{3+3.5}{2}(1) + \frac{3.5+4}{2}(1) + \frac{4+5}{2}(1) = \frac{23}{2} = 11.5$. The assumption $f''(x) \ge 0$ implies convexity of f, which forces the polygonal curve of the Trapezoidal approximation to stay above the curve $y = f(x)$.

30. **(Jars)** Twelve identical straws are distributed at random into three jars. Find the approximate probability that the first jar contains three straws.

[**a**] $18(\frac{2}{3})^{11}$ (b) $19(\frac{2}{3})^{11}$ (c) $20(\frac{2}{3})^{11}$ (d) $21(\frac{2}{3})^{11}$ (e) $22(\frac{2}{3})^{11}$

Solution to 30. The first solution is by counting favorable events. The sample space contains 3^{12} events. If a favorable event occurs, then the remaining 9 straws are distributed among 2 jars. The number of events which forms the jar with three straws is $12!/(3!9!) = (12)(11)(10)/6 = (20)(11)$. The other two jars can be filled in 2^9 ways. So the favorable events amount to $(20)(11)(2^9) = (55)(2^{11})$. The probability is $(55)2^{11}/3^{12}$. The fraction 55/3 is 18.33, therefore the closest answer is $18(\frac{2}{3})^{11}$.

A simpler solution identifies the answer as arising from probabilities $p = \frac{1}{3}$ and $q = \frac{2}{3}$ and the binomial sum for $(p + q)^{12}$. The term that represents the desired probability is the one for p^3q^9, which is $\binom{12}{3} p^3q^9 = \frac{55}{3}(\frac{2}{3})^{11}$.

Common Problems 31-40

31. **(Inequalities)** Find the maximum of $3x + 2y$ subject to the inequalities $x \ge 0$, $y \ge 0$, $x + y \ge 5$ and $2x + y \le 8$.

(a) 10 (b) 13 (c) 14 (d) 15 [**e**] 16

Solution to 31. The function $z = 3x + 2y$ is a plane in R^3. The inequalities describe a scalene triangle in the plane. The maximum occurs at one of the triangle's vertices $(0, 5)$, $(0, 8)$, $(3, 2)$. The values of z are 10, 16, 13, respectively, and the maximum of $3x + 2y$ is 16.

32. **(Range)** Select below a possible value of $\sqrt{D}$, given $D = a^2b + ab$ and integers a and b satisfy $|a - b| = 1$.

(a) $68\sqrt{2}$ (b) $55\sqrt{3}$ (c) $43\sqrt{5}$ [**d**] $50\sqrt{6}$ (e) $55\sqrt{5}$

Solution to 32. Consider first the case $a = x$, $b = x + 1$. Then $D = x^2(x + 1) + x(x + 1) = x^3 + 2x^2 + x = x(x + 1)^2$ and $\sqrt{D} = \sqrt{x}(x + 1)$. The values given range from 91 to 123. Solving $x(x + 1)^2 = 91^2$ and $x(x + 1)^2 = 123^2$ tells us to check values $x = 19$ to $x = 25$. Only $x = 24$ gives a value in the list, which is $50\sqrt{6}$.

The second case $a = x + 1$, $b = x$ results in $D = x(x + 1)^2 + x(x + 1) = x(x + 1)(x + 2)$. The relevant range for x is 21 to 24, which produces no answer from the list.

33. **(Rhombus)** A rhombus of area 20 cm^2 has diagonals the in ratio of 2:1. Find the side in centimeters.

[**a**] 5 (b) 23/5 (c) $2\sqrt{5}$ (d) 14/3 (e) 11/2

Solution to 33. The diagonals of a rhombus are perpendicular, therefore the area of a subtriangle is $(1/2)(2a)(a)$ where the diagonals have lengths $2a$ and $4a$. By the Pythagorean Theorem, $4a^2 + a^2 = x^2$, where x is the unknown side. The area hypothesis gives $4a^2 = 20$. Finally, $x^2 = 25$ and the answer is $x = 5$.

34. **(Clock)** As the hands of a clock move from 3am to 2:59pm, how many times do they form a right angle?

(a) 18 [**b**] 22 (c) 23 (d) 24 (e) None

Solution to 34. This problem can be solved by drawing 24 clock diagrams or by winding a standard watch with hour and minute hands. We explain below why the answer is not 24 and not 23.

If the minute hand and the hour hand form a right angle, then a fixed number of minutes a elapses before the hands can form another right angle. A simple estimate is $a = 32.75$, approximately. The fact $a > 30$ is enough to argue that 24 is the wrong answer. As a check on the logic, $32.75(22) > 720$, which accounts for the full number of minutes $30(24) = 720$ from 3am to 3pm.

In the first hour, 3am to 4am, the hands are at right angles 2 times, once at 3am and then after a minutes. During the four hour period from 4am to 7:59am, the hands are at right angles two times per hour for 4 hours, making 8 times. In the time period, 8am to 8:59pm, they are at right angles only once. The total for five hours and 59 minutes is $2 + 8 + 1 = 11$. The geometry from 3am to 8:59am is the same as 9am to 2:59pm. So the total is $11 + 11 = 22$.

A precise solution can be based upon the angle increase between hands, which is $330°$ per hour. Let t be in hours and let m stand for an integer. For a right angle to occur, $330t$ must be an integral multiple of 180, i.e., $330t = 180m$. The domain of t is $0 \le t \le 11\frac{59}{60}$. Therefore, m satisfies $0 \le m = \frac{330}{180}t \le \frac{330}{180}(11+\frac{59}{60}) = \frac{7909}{360} < 22$. This implies that m can be any integer from 0 to 21, corresponding to 22 times t.

35. **(Dice)** Two dice are tossed 100 times. Find the probability of getting 12 points one time or two times.

(a) $\frac{422}{63}(\frac{35}{36})^{99}$ (b) $\frac{281}{42}(\frac{35}{36})^{99}$ (c) $\frac{47}{7}(\frac{35}{36})^{99}$ [**d**] $\frac{845}{126}(\frac{35}{36})^{99}$ (e) None

Solution to 35. Let $p = \frac{1}{36}$, $q = \frac{35}{36}$. Then $p+q = 1$ and p is the probability of 12 points in one toss. The probability of one or two successful tosses out of the next 100 is the sum of the terms of the binomial expansion $(p+q)^{100}$ with powers pq^{99} and p^2q^{98}. These terms add to

$$\binom{100}{1} p^1 q^{99} + \binom{100}{2} p^2 q^{98}$$

which equals

$$q^{98}(100pq + 50(99)p^2) = q^{98}\frac{4225}{648}.$$

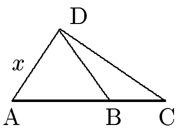
This agrees with $\frac{845}{126}(\frac{35}{36})^{99}$.

36. **(Factorial)** Evaluate the sum
$2(4)(2!)^2 + 3(5)(3!)^2 + 4(6)(4!)^2 + \cdots + 99(101)(99!)^2$.

(a) $(100!)^2$ (b) $(99!)^2 - 4$ (c) $(100!)^2 - 1$ [**d**] $(100!)^2 - 4$ (e) $(99!)^2 - 1$

Solution to 36. This is $\sum_{n=2}^{99} n(n+2)(n!)^2$. Write $n(n+2)(n!)^2 = (n+1)^2(n!)^2 - (n!)^2 = ((n+1)!)^2 - (n!)^2$ and observe that the summation is a telescoping sum. The answer is $(100!)^2 - (2!)^2$.

37. **(Triangle)** In the figure, let triangles ABD and BCD be given with $AB = 2$, $BC = 1$, $AD = x$, $BD = 4 - x$ and $CD = 1 + x$. Find x.



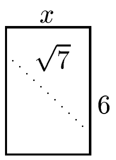
(a) $\frac{27}{14}$ (b) $\frac{93}{50}$ (c) $\frac{64}{35}$ [**d**] $\frac{13}{7}$ (e) None

Solution to 37. The problem can solved with the law of cosines $c^2 = a^2 + b^2 - 2ab\cos C$ or with Heron's formula $A = \sqrt{s(s-a)s-b)s-c)}$, $s = \frac{1}{2}(a+b+c)$.

The trigonometric solution starts with defining angle θ to be angle BCD. Then $2(2)(x)\cos\theta = x^2 + 2^2 - (4-x)^2 = 8x - 12$ and $2(3)(x)\cos\theta = 3^2 + x^2 - (1+x)^2 = 8 - 2x$. Equating cosine terms results in the linear equation $3(8x - 12) = 2(8 - 2x)$, which has solution $x = 13/7$.

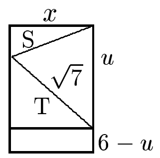
The Heron's formula solution, which uses no trigonometry, proceeds as follows. Drop an altitude $\overline{DE}$ to a point E along $\overline{AB}$ which divides the segment $\overline{AB}$ as $t = AE$ and $2 - t = EB$. Let $h = DE$. The Pythagorean theorem implies $h^2 = x^2 - t^2 = (4-x)^2 - (2-t)^2$, whereby $t = 2x - 3$ and $h^2 = 12x - 3x^2 - 9$. By positivity of t, we have $x > 3/2$. The area α of the largest triangle satisfies $\alpha^2 = (3h/2)^2$ and also $\alpha^2 = s(s-x)s - 1 - x)(s-3)$ where $s = (x+1+x+3)/2$ is the semiperimeter used in Heron's formula. Set the two formulas for α^2 equal to obtain a quadratic equation for x, $7x^2 + 13 - 20x = 0$, with roots $13/7$ and 1. The requirement $x > 3/2$ implies $x = 1$ is invalid. The answer must be $x = 13/7$.

38. **(Fold)** A rectangular piece of paper of height 6 and width $x < 5$ is folded such that two diagonally opposite vertices coincide, causing a crease in the paper of length $\sqrt{7}$. Find x.



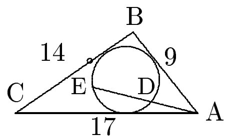
(a) $\sqrt{7}$ [**b**] $\sqrt{6}$ (c) $\frac{7}{2\sqrt{2}}$ (d) $\frac{6}{\sqrt{7}}$ (e) $\frac{2\sqrt{7}}{\sqrt{5}}$

Solution to 38. The right figure does the problem. Add to the given figure to obtain the figure at the right. Using triangle T, we have $x^2 + (2u-6)^2 = 7$. Using triangle S, we have $(6-u)^2 + x^2 = u^2$, because the folding reproduces the dimension u on the hypotenuse of triangle S.



Solving gives $4u^2 - 12u - 7 = 0$ or $u = 7/2$. Then $x^2 + (2u-6)^2 = 7$ implies the result $x^2 = 6$.

39. **(Segment)** The triangle ABC in the figure has sides of 14, 9 and 17 units. A copy of segment $\overline{AB}$ is rotated from vertex A until it meets the inscribed circle at the points D and E. Given $AE = AB$, approximate AD.



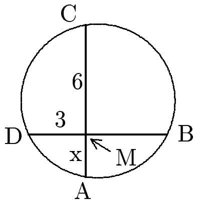
(a) 3.6 (b) 3.8 [**c**] 4.0 (d) 4.2 (e) 4.4

Solution to 39. The idea is to apply

> *If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment equals the product of the measures of the secant segment and its external secant segment.*

The secant segment is $AE = 9$. The tangent segment can be shown to be 6 (see below). Then $6^2 = (AE)(AD)$, which gives $AD = 4$. To establish the tangent segment length of 6, argue as follows. Let the sides of lengths 14, 9 and 17 be divided in ratios $x{:}z$, $z{:}y$ and $y{:}x$, respectively, by the contact points with the inscribed circle. We used: *If two segments from the same exterior point are tangent to a circle, then they are congruent.* Then $x + z = 14$, $y + z = 9$ and $y + x = 17$. Solving these three equations in three unknowns gives $x = 11$, $y = 6$, $z = 3$.

40. **(Diameter)** Let points A, B, C, D be on the circumference of a circle of diameter 10, as in the figure. Assume segments $\overline{AC}$ and $\overline{BD}$ meet orthogonally at M with $AM = x$, $CM = 6$ and $DM = 3$. Find x.



(a) 3 (b) $\frac{7}{2}$ (c) $\frac{2}{5}\sqrt{65}$ (d) $\sqrt{10}$ [**e**] $\sqrt{11}$

Solution to 40. The first step is to establish the measure $BM = 2x$ from the ratio relation $(CM)(AM) = (DM)(BM)$. Locate the center N of the circle by constructing perpendiculars from the midpoints of the two chords $\overline{AC}$ and $\overline{BD}$. Then $\overline{BN}$ is the hypotenuse of a right triangle of legs a and b, calculated as $a = \frac{6+x}{2} - x = 3 - \frac{1}{2}x$ and $b = \frac{1}{2}(BD) == x + \frac{3}{2}$. Since $BN = 10/2$, the Pythagorean theorem $a^2 + b^2 = 5^2$ gives a quadratic equation $(3 - x/2)^2 + (x + 3/2)^2 = 25$ that can be solved for $x = \sqrt{11}$.

1995 Utah State Math Contest Answer Keys

Answers to 1995 Exam Grade 7			
1. (d)	11. (a)	21. (b)	31. (a)
2. (b)	12. (b)	22. (a)	32. (a)
3. (b)	13. (c)	23. (a)	33. (d)
4. (a)	14. (d)	24. (c)	34. (b)
5. (d)	15. (c)	25. (d)	35. (d)
6. (b)	16. (c)	26. (a)	36. (d)
7. (e)	17. (d)	27. (c)	37. (e)
8. (b)	18. (c)	28. (d)	38. (c)
9. (a)	19. (b)	29. (b)	39. (d)
10. (d)	20. (c)	30. (d)	40. (c)

Answers to 1995 Exam Grade 8			
1. (c)	11. (c)	21. (d)	31. (a)
2. (e)	12. (e)	22. (c)	32. (a)
3. (b)	13. (d)	23. (b)	33. (d)
4. (e)	14. (c)	24. (c)	34. (b)
5. (e)	15. (a)	25. (c)	35. (d)
6. (d)	16. (c)	26. (b)	36. (d)
7. (c)	17. (c)	27. (a)	37. (e)
8. (b)	18. (a)	28. (d)	38. (c)
9. (b)	19. (b)	29. (c)	39. (d)
10. (b)	20. (b)	30. (d)	40. (c)

Answers to 1995 Exam Grade 9			
1. (c)	11. (e)	21. (c)	31. (a)
2. (b)	12. (c)	22. (a)	32. (a)
3. (c)	13. (d)	23. (a)	33. (d)
4. (d)	14. (c)	24. (a)	34. (b)
5. (c)	15. (b)	25. (c)	35. (d)
6. (a)	16. (c)	26. (b)	36. (d)
7. (d)	17. (c)	27. (c)	37. (e)
8. (a)	18. (d)	28. (a)	38. (c)
9. (e)	19. (b)	29. (c)	39. (d)
10. (a)	20. (b)	30. (e)	40. (c)

Answers to 1995 Exam Grade 10			
1. (d)	11. (a)	21. (b)	31. (e)
2. (a)	12. (a)	22. (b)	32. (d)
3. (c)	13. (c)	23. (d)	33. (a)
4. (a)	14. (c)	24. (b)	34. (b)
5. (d)	15. (c)	25. (a)	35. (d)
6. (b)	16. (b)	26. (c)	36. (d)
7. (c)	17. (c)	27. (a)	37. (d)
8. (b)	18. (c)	28. (e)	38. (b)
9. (b)	19. (b)	29. (a)	39. (c)
10. (b)	20. (b)	30. (c)	40. (e)

Answers to 1995 Exam Grade 11			
1. (e)	11. (b)	21. (e)	31. (e)
2. (a)	12. (a)	22. (e)	32. (d)
3. (a)	13. (c)	23. (d)	33. (a)
4. (b)	14. (d)	24. (e)	34. (b)
5. (a)	15. (e)	25. (b)	35. (d)
6. (d)	16. (d)	26. (c)	36. (d)
7. (b)	17. (a)	27. (d)	37. (d)
8. (e)	18. (d)	28. (d)	38. (b)
9. (a)	19. (e)	29. (e)	39. (c)
10. (a)	20. (c)	30. (a)	40. (e)

Answers to 1995 Exam Grade 12			
1. (c)	11. (a)	21. (e)	31. (e)
2. (a)	12. (d)	22. (b)	32. (d)
3. (c)	13. (c)	23. (d)	33. (a)
4. (e)	14. (e)	24. (b)	34. (b)
5. (a)	15. (a)	25. (b)	35. (d)
6. (d)	16. (c)	26. (a)	36. (d)
7. (e)	17. (a)	27. (a)	37. (d)
8. (e)	18. (b)	28. (b)	38. (b)
9. (b)	19. (b)	29. (c)	39. (c)
10. (d)	20. (b)	30. (a)	40. (e)