MATH 2200 Learning Celebration #3 Solutions

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General Comments:

• If you’re still having trouble with functions, that’s okay (they are hard!), but know that they will not go away! Analysis, for instance, is all about particular classes of functions (continuous, differentiable, integrable, etc.). There will be plenty of function questions on the final learning celebration, and possibly even on the next two LCs, so I strongly advise you to keep trying to understand proofs involving functions. You are cordially invited to my office for free private tutoring!

• The maximum possible score was 70. The mean was 48.5 and the median was 48. The highest score was 70.

Part A: Computation (10 points)

Use the Euclidean algorithm and back substitution to find \( s, t \in \mathbb{Z} \) such that \( \gcd(56, 21) = s \cdot 56 + t \cdot 21 \).

Solution:

\[
\begin{align*}
56 &= 2 \cdot 21 + 14 \\
21 &= 1 \cdot 14 + [7] \\
14 &= 2 \cdot 7 + 0
\end{align*}
\]

So \( \gcd(56, 21) = 7 \). Then

\[
\begin{align*}
7 &= 21 - 1 \cdot 14 \\
&= 21 - 1 \cdot (56 - 2 \cdot 21) \\
&= 3 \cdot 21 - 1 \cdot 56,
\end{align*}
\]

so \( s = -1 \) and \( t = 3 \).
Part B: Claims (60 points)

Claim 1. Let $A$, $B$, and $C$ be sets, and let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be functions. If $g \circ f$ is surjective, then $g$ is surjective.

Proof. Let $c \in C$ be arbitrary. Since $g \circ f$ is surjective, there is $a \in A$ such that $(g \circ f)(a) = c$, namely $g(f(a)) = c$. Thus $g$ maps $f(a) \in B$ to $c$, so $g$ is surjective.

Comments:

• A crucial step in proving claims about injectivity and surjectivity is to set up your proof correctly. In this claim, you need to prove that $g$ is surjective, so you have to show that each element of the codomain of $g$, namely each $c \in C$, gets mapped to by $g$. You should begin your proof by establishing that $c \in C$ is an arbitrary element, and spend the rest of the proof finding an element of $B$ that gets mapped to $c$ by $g$. At some point you will need to use the assumption that $g \circ f$ is surjective.

Claim 2. Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. If $a \mid b$ and $a \mid c$, then $a \mid (7b - c)$.

Proof. Since $a \mid b$, there is $k \in \mathbb{Z}$ such that $b = ak$. Since $a \mid c$, there is $l \in \mathbb{Z}$ such that $c = al$. Then $7b - c = 7ak - al = a(7k - l)$, so $a \mid (7b - c)$.

Claim 3. The function $\mathbb{R} \xrightarrow{f} \mathbb{R}_{\geq 0}$ with rule $f(x) = x^4$ is bijective.

This claim is false! Since $f(-1) = 1 = f(1)$, $f$ is not injective, so $f$ is not bijective.

Comments:

• You shouldn’t write that you are giving a “counterexample” to this claim because a counterexample is a particular case of a claim in which the assumptions are met and the conclusion fails. But this claim does not involve multiple cases! That’s because this claim does not involve any general sets or functions; the relevant sets and the function $f$ are specifically identified. On the other hand, a claim like “Let $n \in \mathbb{Z}$. Then $n$ is odd” has many cases, one for each choice of $n \in \mathbb{Z}$ (for instance, $n = 2$). In some of those cases, the conclusion is true, but the fact that there is even just one case making the conclusion false, namely a counterexample, means that the claim is false.

• My solution is not a counterexample. It is simply a direct proof that $f$ is not injective.

Claim 4. The function $\mathbb{R}_{>2} \xrightarrow{f} \mathbb{R}_{>0}$ with rule $f(x) = \frac{1}{x-2}$ is bijective.
Proof. Let $\mathbb{R}_{>0} \xrightarrow{g} \mathbb{R}_{>2}$ be the function with rule $g(y) = \frac{1}{y} + 2$. Then $\mathbb{R}_{>2} \xrightarrow{g \circ f} \mathbb{R}_{>2}$ has rule

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x - 2}\right) = x - 2 + 2 = x,$$

so $g \circ f = \text{id}_{\mathbb{R}_{>2}}$. Similarly, $\mathbb{R}_{>0} \xrightarrow{f \circ g} \mathbb{R}_{>0}$ has rule

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{1}{y} + 2\right) = y,$$

so $f \circ g = \text{id}_{\mathbb{R}_{>0}}$. Thus $g$ is the inverse of $f$, so $f$ is bijective. \hfill \square

Comments:

- Be sure to state the domain and range of $g$ when you define it! And don’t call $g$ the “inverse of $f$” until you have proved the compositions are the identity functions!
- Don’t start your proof by saying “If $f$ is bijective, then $f$ has an inverse function...”. You are not allowed to assume $f$ is bijective in this proof, because that’s what you are trying to prove! There’s no point in proving anything as a result of $f$ being bijective, because if $f$ is bijective then you are already done with the proof!

Claim 5. Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. If $a \mid b$ and $b \mid c$, then $c \mid a$.

This claim is false! Counterexample: let $a = 1$, $b = 1$, and $c = 2$. Then $1 \mid 1$ and $1 \mid 2$, but $2 
mid 1$.

Comments:

- Read each claim carefully!
- For this claim, your disproof should take the form of a “counterexample” because the claim has many cases, corresponding to the possible values of $a, b, c \in \mathbb{Z}$. Your counterexample should be a specific choice of $a, b, c \in \mathbb{Z}$ such that $a \neq 0$, $b \neq 0$, $a \mid b$, $b \mid c$, and $c \nmid a$. There are many possible counterexamples!

Claim 6. Let $A$, $B$, and $C$ be sets, and let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be functions. If $f$ is injective, then $g \circ f$ is injective.

This claim is false! Let $A = \{\alpha, \beta\}$, $B = \{\gamma, \delta\}$, and $C = \{\epsilon\}$. Let $A \xrightarrow{f} B$ be the function $f(\alpha) = \gamma$, $f(\beta) = \delta$, which is injective. Let $B \xrightarrow{g} C$ be the function $g(\gamma) = g(\delta) = \epsilon$. Then $A \xrightarrow{g \circ f} C$ is the function $(g \circ f)(\alpha) = (g \circ f)(\beta) = \epsilon$, which is not injective.