General Comments:

- Sketch the graph of the function before starting a proof! (Scratch work, naturally.)

- State the claims properly! “The function \( Z \xrightarrow{f} Z \) with rule \( f(n) = 4 - n \)” is not a statement! See how I’ve stated the claims below.

- To prove a function is not bijective, argue that it is not injective or not surjective. To prove a function is bijective, prove that it has an inverse function.

- The symbols \((\subseteq)\) and \((\supseteq)\) can be used in proofs that two sets are equal. The symbols \((\implies)\) and \((\iff)\) can be used in proofs of “if and only if” claims. These symbols do not make sense in other contexts! They have very specific meanings and are not just arbitrary symbols used to label multiple parts of a proof.

- Defining a rule by \( g(y) = \frac{1}{y} \) doesn’t make sense! The output \( \frac{1}{y} \) has to be written in terms of the input variable \( y \). How do I know what \( g(3) \) should be when the rule refers to some strange alien species \( x \) that I’ve never encountered?

- Don’t give variable names to integers or real numbers unless you have to. For instance, if you want to say something about the value of \( f(x) \) when \( x = 2 \), then simply refer to \( f(2) \) instead of saying “when \( x = 2 \), then \( f(x) \) ...”. Likewise, don’t redefine functions that are already defined. If the inverse of \( f \) is equal to \( f \), then talk about \( f \) being its own inverse. See below.

- Please read all the proofs, disproofs, and comments below! If you have any questions, please ask.

\section*{Exercise 3.5.13}

\textbf{Claim} (a). The function \( Z \xrightarrow{f} Z \) with rule \( f(n) = 4 - n \) is bijective.

\textit{Proof.} Since \( Z \xrightarrow{f \circ f} Z \) has rule \( (f \circ f)(n) = f(f(n)) = 4 - (4 - n) = n \), we see that \( f \circ f = \text{id}_Z \) and \( f \) is its own inverse. Since \( f \) has an inverse, it is bijective. \( \square \)

Comments:
• When you are defining a function that will turn out to be the inverse, don’t say “Let $g$
be the inverse of $f$”. You can’t say that because you don’t know that $f$ has an inverse –
that is what you are trying to prove!

• Whenever you define a new function, you must specify the domain and codomain!
Otherwise you haven’t defined a function.

• Don’t show the process by which you computed the inverse function (solving for $m =
4 - n$ for $n$). That’s scratch work!

Claim (b). The function $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$ with rule $f(n) = n^3$ is bijective.

This claim is false! Since 2 is not the cube of an integer, $f$ is not surjective. Thus $f$
is not bijective.

Comments:

• To prove that a function is not bijective, you should argue that it is not injective or
that it is not surjective. This is better than arguing that the rule you get when you
try to solve $m = f(n)$ for $n$ is not an inverse function, because it is a bit difficult to
rigorously argue that there cannot be some other inverse.

• Proving that $f$ is not surjective is not a “counterexample”. It is just an argument that
$f$ is not surjective. A counterexample only makes sense for a general claim: it is a
choice of values for the general objects in the claim that makes the assumptions true
and the conclusion false. Here the claim is about a specific function.

Claim (c). The function $\mathbb{R} \xrightarrow{f} \mathbb{R}_{\geq 1}$ with rule $f(x) = x^2 + 1$ is bijective.

This claim is false! Since $f(1) = 2 = f(-1)$, $f$ is not injective. Thus $f$ is not bijective.

Comments:

• Many of you “proved” that the function $\mathbb{R}_{\geq 1} \xrightarrow{g} \mathbb{R}$ with rule $g(y) = \sqrt{y - 1}$ is the
inverse of $f$. It is instructive to figure out what goes wrong when you are checking the
compositions. Try it! It should be clear from the graph of $f$ that $f$ is not injective,
and therefore cannot have an inverse.

Claim (d). The function $\mathbb{R}_{>0} \xrightarrow{f} \mathbb{R}_{>0}$ with rule $f(x) = \frac{1}{x}$ is bijective.

Proof. Since $\mathbb{R}_{>0} \xrightarrow{f \circ f} \mathbb{R}_{>0}$ has rule $(f \circ f)(x) = f(f(x)) = f(\frac{1}{x}) = x$, we see that $f \circ f =
id_{\mathbb{R}_{>0}}$. Thus $f$ is its own inverse, so $f$ is bijective.

Comments:
• It was silly of me to assign two exercises, namely (a) and (d), featuring functions that are their own inverses. Most invertible functions are not their own inverses!

Claim (e). The function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ with rule $f(x) = \sin(x)$ is bijective.

This claim is false! Since $f(x) \leq 1$ for all $x \in \mathbb{R}$, we see that $f$ is not surjective. Thus $f$ is not bijective.

Comments:

• It is even easier to argue that $f$ is not injective, but I decided to argue non-surjectivity for variety’s sake!

Claim (f). The function $[-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{f} [-1, 1]$ with rule $f(x) = \sin(x)$ is bijective.

Proof. Consider the function $[-1, 1] \xrightarrow{g} [-\frac{\pi}{2}, \frac{\pi}{2}]$ with rule $g(y) = \arcsin y$. The rule arcsin is well-defined on the domain $[-1, 1]$ and takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $g$ is a function. Since $[-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{gf} [-\frac{\pi}{2}, \frac{\pi}{2}]$ has rule $(g \circ f)(x) = g(f(x)) = \arcsin(\sin x) = x$, and $[-1, 1] \xrightarrow{fg} [-1, 1]$ has rule $(f \circ g)(y) = f(g(y)) = \sin(\arcsin y) = y$, we see that $g \circ f = \text{id}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ and $f \circ g = \text{id}_{[-1, 1]}$. Thus $g$ is the inverse of $f$, so $f$ is bijective. \[\square\]

Comments:

• The purpose of this claim was to get you thinking about domains and codomains and apply what you learned about inverse functions from trigonometry class. In trig, you go to great pains to find the right domain and codomain for arcsin – hopefully you now have an appreciation for why that’s so important.

• This argument is far from rigorous! We haven’t even defined the sine function in class, so we have no chance of writing a rigorous proof involving sine. A nice way to define sine is via the power series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

but that requires a good understanding of convergence of infinite sums. Take Foundations of Analysis!