General Comments:

• Make sure you can draw and interpret Venn diagrams. (Scratch work, of course.)

• To prove that two sets are equal, you should almost always prove both containments, namely \((\subseteq)\) and \((\supseteq)\).

• Remember that \(|A|\) means the cardinality (number of elements) of \(A\), which is a number, not a set!

• Labeling is a decent way to prove facts about cardinality of finite sets, but it is rarely a good way of proving that sets are equal. For instance, general sets can be infinite, so it’s hard to pick “general” labels. In fact, some sets (such as \(\mathbb{R}\)) are so big that it is impossible to put their elements in a list!

• The empty set is denoted \(\emptyset\). This is not the same as \(\{\emptyset\}\), which is a set containing the empty set as an element!

• Many of the claims below can be proved by contraposition or contradiction, but direct proofs usually work just fine.

Exercise 3.1.12

Claim (a). Let \(A\) and \(B\) be finite sets. Then \(|A \cup B| = |A| + |B|\).

This claim is false! Let \(A = \{\omega\}\) and \(B = \{\omega\}\). Then \(A \cup B = \{\omega\}\), so \(|A \cup B| = 1\) but \(|A| + |B| = 2|\).

Claim (b). Let \(A\) and \(B\) be finite sets. Then \(|A - B| = |A| - |B|\).

This claim is false! Let \(A = \emptyset\) and \(B = \{\text{Carlsen}\}\). Then \(A - B = \emptyset\), so \(|A - B| = 0\) but \(|A| - |B| = -1|\).

Claim (c). Let \(A\) and \(B\) be finite sets. If \(A\) and \(B\) are disjoint, then \(|A \sqcup B| = |A| + |B|\).
Proof. Write $A = \{a_1, a_2, \ldots, a_{|A|}\}$ and $B = \{b_1, b_2, \ldots, b_{|B|}\}$. Since $A \cap B = \emptyset$, none of the $a_i$'s are equal to any of the $b_j$'s, so $A \cup B = \{a_1, a_2, \ldots, a_{|A|}, b_1, b_2, \ldots, b_{|B|}\}$. Thus $|A \cup B| = |A| + |B|$.

Comments:

- This is the only proof for which I think labeling is the best choice. In general, try to avoid labeling.

Claim (d). Let $A$ and $B$ be finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.  

Proof. Note that $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$. Thus by (c),

$$|A \cup B| = |A - B| + |A \cap B| + |B - A|.$$ 

Also note that $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$, so that by (c),

$$|A| + |B| - |A \cap B| = (|A - B| + |A \cap B|) + (|B| + |A \cap B|) - |A \cap B|$$

$$= |A - B| + |A \cap B| + |B - A|.$$ 

Thus $|A \cup B| = |A| + |B| - |A \cap B|$. □

Comments:

- This was not a very fair problem because we haven’t proved the identities I cited in the proof above (such as $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$). But you should be able to prove those identities!

Exercise 3.1.15

Claim (a). Let $A$, $B$, and $C$ be sets. Then $A \subseteq B$ if and only if $A \cap B = A$.

Proof.

$(\Rightarrow)$: (⊆): If $a \in A \cap B$, then $a \in A$.  

$(\supseteq)$: If $a \in A$, then $a \in B$ since $A \subseteq B$. Thus $a \in A$ and $a \in B$, so $a \in A \cap B$.  

$(\iff)$: If $a \in A$, then $a \in A \cap B$ since $A = A \cap B$. Thus $a \in B$. □

Comments:

- The main difficulties in this proof are knowing exactly what you have to prove and keeping track of what you are doing.

Claim (b). Let $A$, $B$, and $C$ be sets. Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Proof. ($\subseteq$): If $a \in A \cup (B \cap C)$, then $a \in A$ or $a \in B \cap C$. In the case $a \in A$, then $a \in A \cup B$ and $a \in A \cup C$, thus $a \in (A \cup B) \cap (A \cup C)$. If instead $a \in B \cap C$, then $a \in B$ and $a \in C$, thus $a \in A \cup B$ and $a \in A \cup C$, so $a \in (A \cup B) \cap (A \cup C)$.

($\supseteq$): If $a \in (A \cup B) \cap (A \cup C)$, then $a \in A \cup B$ and $a \in A \cup C$. In the case $a \in A$, then $a \in A \cup (B \cap C)$. If instead $a \notin A$, then $a \in B$ and $a \in C$, namely $a \in B \cap C$, so $a \in A \cup (B \cap C)$.

\[\blacksquare\]

Claim (c). Let $A$ and $B$ be sets and let $X$ be a set containing $A$ and $B$. Then $X - (A \cup B) = (X - A) \cap (X - B)$.

Proof. ($\subseteq$): If $a \in X - (A \cup B)$, then $a \in X$ and $a \notin A \cup B$, namely $a \notin A$ and $a \notin B$. Thus $a \in X - A$ and $a \in X - B$, so $a \in (X - A) \cap (X - B)$.

($\supseteq$): If $a \in (X - A) \cap (X - B)$, then $a \in X - A$ and $a \in X - B$. Thus $a \in X$, $a \notin A$, and $a \notin B$. Therefore $a \notin A \cup B$, so $a \in X - (A \cup B)$.

\[\blacksquare\]

Comments:
- Actually, claims (c) and (d) are still true if we drop the assumption that $X$ contains $A$ and $B$. The exact same proof works! It’s better not to have unnecessary assumptions in a claim, so I erred!

Claim (d). Let $A$ and $B$ be sets and let $X$ be a set containing $A$ and $B$. Then $X - (A \cap B) = (X - A) \cup (X - B)$.

Proof. ($\subseteq$): If $a \in X - (A \cap B)$, then $a \in X$ and $a \notin A \cap B$. Thus $a \notin A$ or $a \notin B$, so $a \notin X - A$ or $a \notin X - B$, namely $a \in (X - A) \cup (X - B)$.

($\supseteq$): If $a \in (X - A) \cup (X - B)$, then $a \in X - A$ or $a \in X - B$. So $a \in X$ and also $a \notin A$ or $a \notin B$. Thus $a \notin A \cap B$, so $a \in X - (A \cap B)$.

\[\blacksquare\]

Claim (e). Let $A$, $B$, and $C$ be sets. If $A \cup C = B \cup C$, then $A = B$.

This claim is false! Counterexample: Let $A = \emptyset$, $B = \{\zeta\}$, and $C = \{\zeta\}$. Then $A \cup C = \{\zeta\} = B \cup C$, but $A \neq B$.

Claim (f). Let $A$, $B$, and $C$ be sets. If $A \cap C = B \cap C$, then $A = B$.

This claim is false! Counterexample: Let $A = \emptyset$, $B = \{pegasus\}$, and $C = \emptyset$. Then $A \cap C = \emptyset = B \cap C$, but $A \neq B$.

Claim (g). Let $A$, $B$, and $C$ be sets. If $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.

Proof. ($\supseteq$): If $a \in A$, then $a \in A \cup C$, so $a \in B \cup C$. If $a \in B$, then we are done. Otherwise $a \in C$, so $a \in A \cap C$. But then $a \in B \cap C$, so $a \in B$.

($\subseteq$): Since the claim is completely symmetric in $A$ and $B$, this part of the proof is the same as ($\supseteq$) with the roles of $A$ and $B$ reversed.

\[\blacksquare\]
Comments:

- Remember: you are trying to prove that $A = B$, and you will use the assumptions $A \cup C = B \cup C$ and $A \cap C = B \cap C$ to do so. Do not start by assuming $a \in A \cup C$ or $a \in A \cap C$. To prove a statement like $A = B$, you should be showing that $A \subseteq B$ and that $A \supseteq B$. This claim can also be proved nicely by contradiction.