Strange Duality and Quot Schemes for Del Pezzo Surfaces
Aaron Bertram, Thomas Goller, and Drew Johnson
University of Utah University of Oregon

Notation
• Let $S$ be a del Pezzo surface over $\mathbb{C}$, namely $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a blowup of $\mathbb{P}^2$ at $\leq 8$ general points.
• Let $e, f \in H^*(S, \mathbb{Q})$ be orthogonal with respect to the Mukai pairing $\langle e, f \rangle$.

Strange duality
• Let $M(e^\vee), M(f)$ be the moduli spaces of semistable sheaves on $S$ with Chern characters $e^\vee, f$.
• Assume the jumping locus $\Theta = \{ (E, F) \mid h^0(E \otimes F) > 0 \} \subset M(e^\vee) \times M(f)$ has the structure of a Cartier divisor.
• Let $\Theta_f \subset M(e^\vee)$ and $\Theta_e \subset M(f)$ be general restrictions of $\Theta$ to fibers of $pr_1, pr_2$. Then
  $$H^0(\Theta_f) = H^0(\Theta_f) \otimes H^0(\Theta_e).$$
and thus
  $$\dim H^0(\Theta_f) = H^0(\Theta_f) \otimes H^0(\Theta_e).$$
• A section defining $\Theta$ determines a pairing
  $$SD_{e,f}: H^0(\Theta_f) \otimes H^0(\Theta_e) \rightarrow \mathbb{C}.$$

Le Potier’s strange duality conjecture: $SD_{e,f}$ is a perfect pairing.

Finite quot schemes
• Marian and Oprea used the following idea to prove strange duality for curves [4].
• Let $V$ be a vector bundle with $ch(V) = e + f$.
• By the orthogonality of $e$ and $f$, the quotient $\text{Quot}(V, f)$ is expected to be finite and reduced.
• Suppose each point of the quotient scheme
  $$0 \rightarrow E_i \rightarrow V \rightarrow F_i \rightarrow 0$$
has $E_i, F_i$ semistable and $\text{Ext}^1(E_i, F_i) = 0$ for all $i$. By stability, $\text{Hom}(E_i, F_i) > 0$ for $i \neq j$.
• Let $H_E \in H^0(\Theta(e^\vee))$ and $H_F \in H^0(\Theta(f))$ be hyperplanes of sections vanishing on $E_i$ and $F_i$, respectively.
• $SD_{e,f}$ is non-degenerate on $\langle H_E \rangle$ and $\langle H_F \rangle$ since
  $$SD_{e,f}(H_E, H_F) = \Theta(E_i^*, F_j^*) = 0 \Leftrightarrow i \neq j.$$

Outline for proving strange duality using quotient schemes
(a) Compute dim $H^0(\Theta_f)$, dim $H^0(\Theta_e)$.
(b) Construct $V$ such that $\text{Quot}(V, f)$ is finite and reduced.
(c) Enumerate the points of $\text{Quot}(V, f)$.

(c) Enumeration
On any $S$, let $f = (1, 0, -n)$ and $e = (r, -\lambda, s)$.

Existence theorem
Suppose $n \geq 1$, $r \geq 2$, and $\lambda \gg 0$. Let $V$ be a general stable vector bundle on $\mathbb{P}^2$ with $ch(V) = e + f$. Then $\text{Quot}(V, f)$ is finite and reduced.

(b) Construction
On $\mathbb{P}^2$, let $f = (1, 0, -n)$ and $e = (r, -\lambda, s)$.

Dimension theorem [2]
$$\sum_{k \geq 0} \chi(\Theta_f) z^k = g(z)^{\chi(L)} \cdot f(z)^{\chi(O(1)/2)} \cdot A_{1}(z)^{1-K-K/2} \cdot B_{1}(z)^{K^2},$$
for power series $g, f, A_1, B_1$ in $z$ that depend only on $r$ (and $f$ have closed formulas).

Question. Are there such power series for other moduli spaces?

Synthesis corollary
On $\mathbb{P}^2$, let $f = (1, 0, -n)$, $e = (r, -\lambda, s)$, and $V$ be a general vector bundle with $ch(V) = e + f$. Then the rank of $SD_{e,f}$ is bounded below by the number of points of $\text{Quot}(V, f)$. Moreover,
1. In the “classical” cases $n = 1$ and $(n, r) \in \{ (2, 2), (2, 3), (3, 2) \}$, $SD_{e,f}$ has full rank;
2. When $n \leq 7$, $SD_{e,f}$ is predicted to have full rank.

Further information
Details can be found on the arXiv.

Contact information
• Web: www.math.utah.edu/~goller
• Email: goller@math.utah.edu

References