Research Statement
Thomas Goller

I study quot schemes on algebraic varieties using algebro-geometric and topological techniques. Quot schemes were invented by Grothendieck ([Gro95]), who observed that certain parameter spaces of objects on schemes could themselves be viewed as schemes. Let $V$ be a vector bundle on a complex projective variety $X$ and $\rho$ be a cohomology class on $X$. The quot scheme $\text{Quot}(V, \rho)$ is a projective scheme that parametrizes all quotients $V \to F$ in which $F$ is a vector bundle (more generally, a coherent sheaf) with Chern character $\rho$. If $X$ is a point, then $V$ is a vector space, $\rho$ is a rank, and $\text{Quot}(V, \rho) = \text{Gr}(V, \rho)$ is the Grassmannian of $\rho$-dimensional quotients of $V$. Quot schemes also yield Hilbert schemes, which parametrize all subschemes of $X$ with a particular Hilbert polynomial, when $V$ is the trivial line bundle $\mathcal{O}_X$. Although quot schemes were originally used as a tool to construct other moduli spaces, such as the Gieseker moduli of stable sheaves, they have since become an object of interest in their own right.

On an algebraic curve $C$, a Chern character is an ordered pair $\rho = (s, d)$ of a rank $s$ and a degree $d$. When $V$ is a trivial vector bundle of rank $r$, the quot scheme $\text{Quot}(V, \rho)$ is a compactification of the space of degree-$d$ maps from $C$ to the Grassmannian $\text{Gr}(\mathbb{C}^r, s)$; this point of view can be used to define the Gromov-Witten numbers, which are counts of maps from $\mathbb{P}^1$ to the Grassmannian. Now let the vector bundle $V$ be general of rank $r$. The fiber of a vector bundle quotient $V \to F$ at a point on $C$ is an $s$-dimensional quotient of the fiber of $V$, so choosing a point on $C$ yields a rational map from the quot scheme to the Grassmannian. I am currently working on proving that when $V$, $d$, and the number of points on $C$ are allowed to vary, the cohomology classes of the images of the maps from quot schemes to Grassmannians are compatible with degenerations of the curve, namely they can be assembled into a two-dimensional weighted topological quantum field theory (wTQFT), where the weight is the degree of $V$. When the curve is $\mathbb{P}^1$ and $V$ is a trivial bundle, the classes of the images of the quot schemes are determined by the Gromov-Witten numbers, so the degree-0 part of the wTQFT is the TQFT built from the small quantum cohomology of the Grassmannian.

By cutting up a topological surface into pairs of pants (degenerating the algebraic curve into a rational nodal curve), a TQFT acts as a machine for computing positive-genus data from data on $\mathbb{P}^1$. In particular, the wTQFT can be used to count the points of finite quot schemes on curves, which are closely related to the Verlinde numbers that play a critical role in the strange duality theorem on curves ([MO07], [Bel09]). The moral is that the theory of degenerations of quot schemes on curves is a natural way to compute the Verlinde numbers. A motivating question is whether this is also true of the analogs of the Verlinde numbers in higher dimensions.

When $S$ is a surface, following the ideas of [MO07], I have studied how finite quot schemes can be used to prove Le Potier’s strange duality conjecture on surfaces. Marian and Oprea used other methods to prove strange duality for general K3 surfaces ([MO13]) and in some cases on abelian surfaces ([MO08a], [MO14]), but on other surfaces only a few scattered examples are known ([MO08b], [Dan02], [Yua12], [Yua16]). The analogs of the Verlinde numbers have not been computed in general; the main exception is a power series in [EGL01] that assembles these numbers in the case when $\rho$ is the Chern character of an ideal sheaf of points. In [BGJ16], for this choice of $\rho$, we use finite quot schemes to obtain evidence of strange duality for a large class of examples on del Pezzo surfaces. One key issue is whether finite quot schemes exist, which we answer affirmatively on $\mathbb{P}^2$. Another key issue is how to count the points of these finite quot schemes. Our approach in [BGJ16] uses topological multiple point formulas that agree with the power series [EGL01] but require genericity assumptions we cannot check in our algebraic setting. In analogy with the wTQFT, I plan to construct a theory of degenerations of quot schemes on surfaces that both enumerates finite quot schemes and computes the analogs of the Verlinde numbers, thereby proving strange duality.
Weighted topological quantum field theory

The case of curves is an inspiring example of how quot schemes can be studied using degenerations. Let $C$ be a curve of genus $g$ and $V$ be a general semistable vector bundle on $C$ with rank $r$ and degree $e$. Given general points $p_1, \ldots, p_N$ on $C$ and any non-negative integer $d$, restricting vector bundle quotients $V \to F$ to fibers at each $p_i$ gives a rational map

$$\text{res}_{p_1, \ldots, p_N} : \text{Quot}(V, (s, d)) \to G^N$$

into the $N$-fold Cartesian product of the Grassmannian $G = \text{Gr}(C^r, s)$. By the Künneth formula, the cohomology ring of $G^N$ is isomorphic to $H^*(G, \mathbb{C})^{\otimes N}$. Let $\eta_{g, e, N} \in H^*(G, \mathbb{C})^{\otimes N}$ denote the sum of the classes of the closures of the images of $\text{res}_{p_1, \ldots, p_N}$ for all $d$ such that the map is generically finite (this condition ensures that only finitely many $d$ contribute and that the image records the dimension of the quot scheme). Choosing a partition $N = m + n$ and identifying the first $m$ copies of $H^*(G, \mathbb{C})$ with their dual by the Poincaré pairing, the class $\eta_{g, e, m+n}$ determines a linear map

$$F(g|e)_{m}^{n} : H^*(G, \mathbb{C})^{\otimes m} \to H^*(G, \mathbb{C})^{\otimes n}.$$ 

**Work in progress.** The $F(g|e)_{m}^{n}$ have the structure of a two-dimensional weighted topological quantum field theory that generalizes the quantum cohomology of the Grassmannian.

A two-dimensional TQFT on a vector space $W$ is a functor from the category whose objects are finite unions of oriented circles and whose morphisms are cobordisms into the category of vector spaces, such that the image of the union of $n$ circles is $W^{\otimes n}$. Each cobordism is a topological genus $g$ surface that has $m$ boundary circles with negative orientation and $n$ boundary circles with positive orientation, to which the TQFT associates a linear map $W^{\otimes m} \to W^{\otimes n}$. Gluing positive circles on one topological surface to negative circles on another surface corresponds to (partial) composition of the linear maps, as in Figure 1. The extra data of a *weight* is an integer $e$ attached to each cobordism, which is additive under composition. The key step I am pursuing is showing that the $F(g|e)_{m}^{n}$ satisfy these composition rules. The idea is that given a smooth curve $C$ and a vector bundle $V$ of rank $r$ and degree $e$, there is a degeneration of $C$ into a nodal curve such that $V$ splits into vector bundles of rank $r$ on the components. The sum of the degrees of these vector bundles is $e$, and the quot schemes on the components can be used to recover the quot scheme on $C$ by imposing gluing conditions at the nodes.

![Figure 1: Gluing in the wTQFT](image-url)

---

2
By cutting weighted topological surfaces into pairs of pants and cylinders of weight 1, we see that the wTQFT is determined by the quantum product $F(0|0)^1_1$, the Poincaré pairing $F(0|0)^0_0$, and the weight-shifting operator $F(0|1)^1_1$, which is quantum multiplication by a particular Schubert class. Thus the weight-0 theory, which is equivalent to a Frobenius algebra structure on $H^*(G, \mathbb{C})$, coincides with the TQFT built from the small quantum cohomology of the Grassmannian ([Ber97]). This wTQFT also contains a representation-theoretic TQFT designed by Witten ([Wit95], [MO10]) to compute the Verlinde numbers, which are ranks of vector bundles of conformal blocks on moduli spaces of curves. The wTQFT interprets each of these ranks as the number of points of the finite quot scheme $\text{Quot}(V, (s, d))$ for $e = d = s(g - 1)$ and reproduces the Verlinde formula expressing these numbers as a sum of $(g - 1)st$ powers of the eigenvalues of $F(1|s)^1_1$.

Despite being the simplest quot schemes, finite quot schemes on curves and surfaces are particularly interesting from the point of view of strange duality, as I will discuss in the next section. The wTQFT exhibits finite quot schemes in the context of all quot schemes on curves; this idea might generalize to higher-dimensional varieties.

**Problem 1.** Develop a theory of degenerations of quot schemes on surfaces, modeled on the wTQFT for curves, which counts the points of finite quot schemes.

Constructing such a theory will require understanding degenerations of surfaces, such as the technique of deformation to the normal cone ([Ful98]). One complication is that the components of a singular surface obtained by degeneration will intersect along curves, so the analog of the Grassmannian $\text{Gr}(\mathbb{C}^r, s)$ will be a quot scheme on a curve.

**Strange duality and quot schemes**

**Strange duality on curves.** When $C$ is a curve, Marian and Oprea used finite quot schemes to prove the *strange duality theorem for curves* ([MO07]). Let $\sigma$ and $\rho$ be cohomology classes on $C$ that are orthogonal under the Mukai pairing. There are moduli spaces of coherent sheaves on $C$ associated to $\sigma$ and to $\rho$, and these moduli spaces are each equipped with a determinant line bundle induced by the orthogonal class. There is a natural bilinear pairing

$$\text{SD}_{\sigma, \rho} : \Gamma^\rho_{\sigma} \otimes \Gamma^\rho_{\rho} \to \mathbb{C},$$

called the *strange duality pairing*, between the duals of the vector spaces of sections of these determinant line bundles. The strange duality theorem asserts that $\text{SD}_{\sigma, \rho}$ is a perfect pairing.

Let $V$ be a vector bundle with Chern character $\sigma + \rho$. Each quotient in $\text{Quot}(V, \rho)$ can be extended to a short exact sequence $0 \to E \to V \to F \to 0$ in which $E$ has Chern character $\sigma$. In the case when the quot scheme is finite and reduced, the kernels $E_i$ and quotients $F_i$ yield subspaces of $\Gamma^\rho_{\sigma}$ and $\Gamma^\rho_{\rho}$ on which $\text{SD}_{\sigma, \rho}$ is non-degenerate, and the dimension of these subspaces is equal to the number of points of the quot scheme. Thus the key ingredients in a proof of strange duality are

(a) compute the dimensions of the spaces of sections $\Gamma^\rho_{\sigma}$ and $\Gamma^\rho_{\rho}$,

(b) construct a $V$ with Chern character $\sigma + \rho$ such that $\text{Quot}(V, \rho)$ is finite and reduced, and

(c) enumerate the points of $\text{Quot}(V, \rho)$.

Marian and Oprea perform (b) and (c) and observe that the number of points of $\text{Quot}(V, \rho)$ agrees with the dimensions of $\Gamma^\rho_{\sigma}$ and $\Gamma^\rho_{\rho}$ (which had already been computed using the Verlinde formula), thus proving that the strange duality pairing is non-degenerate and hence perfect.
Strange duality on surfaces. I am particularly interested in using Marian and Oprea’s idea to investigate Le Potier’s strange duality conjecture for surfaces. On a surface $S$, the same outline for proving strange duality applies, but each step becomes more difficult. The first complication is that the Chern character of a coherent sheaf is a triple $(r, D, d)$ of a rank $r$, a divisor class $D$ on $S$, and a number $d$ encoding the second Chern class of the sheaf that is often hard to control.

No general methods for performing (a) are known. A key case where half of (a) can be completed is when $\rho = (1, 0, -n)$ is the Chern character of an ideal sheaf of $n$ points and the moduli space is the Hilbert scheme of points $S[n]$. Let $\sigma = (r, D, d)$ be a general class orthogonal to $(1, 0, -n)$, where $r$ and $D$ are arbitrary and $d$ is determined by orthogonality. In this setting, [EGL01] provide a power series that computes $\dim \Gamma^\sigma_{\rho}$ on any surface $S$:

$$\sum_{n \geq 0} \dim \Gamma^\sigma_{\rho} \cdot z^n = g_r(z) \chi(D) \cdot f_r(z) \frac{1}{2} \chi(O_S) \cdot A_r(z) D, K_S - \frac{1}{2} K_S^2 \cdot B_r(z) K_S^2.$$ 

Here $A_r, B_r, f_r, g_r$ are power series in $z$ whose coefficients depend only on $r$. The power series $f_r$ and $g_r$ have known closed formulas, but $A_r$ and $B_r$ have to be computed using methods such as localization. The exponents are determined by the Euler characteristic $\chi(O_S)$ and by the intersection theory of the divisor $D$ and the canonical divisor $K_S$ of the surface.

This amazing formula, which can be viewed as an analog of the Verlinde formula on curves, is proved by studying the class of $S[n]$ in the complex cobordism ring. The cobordism ring could be the setting in which to develop a theory of degenerations of moduli spaces that extends (a) beyond Hilbert schemes. A related approach involving $K$-theoretic Donaldson invariants and four-dimensional TQFTs has been used in [GNY09] and [Gö16] to deduce some formulas on surfaces. In any case, [EGL01] suggests that we should expect answers to (a) obtained by degeneration methods to be power series rather than sums of powers of eigenvalues as in the case of curves.

**Problem 2.** Develop a theory of degenerations that produces power series for computing the dimension of the space of sections $\Gamma^\sigma_{\rho}$ for other cohomology classes $\rho$.

Moving toward (b) and (c), suppose Quot$(V, \rho)$ is finite and reduced. Each point of the quot scheme exhibits $V$ as an extension $0 \to E \to V \to F \to 0$ in which $\text{Ext}^1(E, F) = 0$. Since the extension is not split, its class in $\text{Ext}^1(F, E) \simeq \text{Ext}^1(E, F \otimes \omega_S)$ must be nonzero. This forces the canonical divisor $K_S$ to be anti-ample, namely $S$ is a del Pezzo surface: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow-up of $\mathbb{P}^2$ at up to 8 general points. In [BGG16], we perform (b) on $\mathbb{P}^2$ when $\rho = (1, 0, -n)$ and $\sigma = (r, -\lambda \text{[line]}, d)$ is a general class orthogonal to $\rho$.

**Theorem** ([BGG16]). Suppose $n \geq 1$, $r \geq 2$, and $\lambda \gg 0$. Let $V$ be a general stable vector bundle on $\mathbb{P}^2$ with Chern character $\sigma + \rho$. Then Quot $(V, \rho)$ is finite and reduced.

The proof uses an incidence variety and dimension counts, which depend on a result of [CHW14] on $\mathbb{P}^2$ that guarantees that the duals of the general stable vector bundles $V$ in the theorem have resolutions of the form

$$0 \to O(-2)^C \to O(-1)^B \oplus O^A \to V^* \to 0.$$ 

By working with a space of such resolutions instead of the moduli space, we replace difficult questions of stability by computations in sheaf cohomology. We focused on the Hilbert scheme in [BGG16], but resolutions involving exceptional vector bundles exist for general $\rho$ on $\mathbb{P}^2$ and on del Pezzo surfaces.

**Problem 3.** Adapt the proof of the theorem for general del Pezzo surfaces using resolutions obtained from strong exceptional collections.
Also in [BGJ16], in the case when $S$ is del Pezzo, $\rho = (1, 0, -n)$, $\sigma = (r, -D, d)$, and $n \leq 7$, we use multiple point formulas to obtain an expected value for the number of points of Quot($V, \rho$) that agrees with the power series from [EGL01].

**Theorem** ([BGJ16]). For $n \leq 7$ and $D$ sufficiently ample, $\dim \Gamma_{\rho}^n$ agrees with the expected number of points of Quot($V, \rho$) for a vector bundle $V$ on $S$ of class $\sigma + \rho$.

To prove the theorem, we exploit the fact that ideal sheaf quotients $V \to \mathcal{I}_Z$, where $Z$ is a subscheme of $S$, are dual to sections of $V^*$ vanishing along $Z$. The fiber of the kernel in the exact sequence

$$0 \to M \to H^0(V^*) \otimes \mathcal{O}_S \to V^* \to 0$$

at a point $p$ collects the sections of $V^*$ that vanish at $p$. Thus the $n$-fold points of the map

$$\phi: P(M) \to P(H^0(V^*))$$

of projective spaces of lines correspond to sections of $V^*$ that vanish at $n$ points, which in turn dualize to quotients $V \to \mathcal{I}_Z$ in which $Z$ consists of $n$ points. Under some topological genericty conditions on $\phi$, the number of $n$-fold points is finite and can be computed for $n \leq 7$ by a multiple point formula ([MR10], [Kaz03]). Thus in the case when $\phi$ is sufficiently general and the quot scheme is reduced, the multiple point formula counts Quot($V, \rho$), hence proving that the strange duality pairing has full rank ($\dim \Gamma_{\rho}^n$ could be larger than $\dim \Gamma_{\rho}^\sigma$). Unfortunately, the topological genericty conditions on $\phi$ are nearly impossible to check (even for maps satisfying algebraic generticity conditions), so multiple point formulas are ineffective in producing a precise answer to (c). Instead, the degeneration methods proposed for studying Problem 1 could yield a more rigorous way to count finite quot schemes on surfaces.\footnote{This alternate method of enumerating finite quot schemes could provide evidence that the multiple point formulas are valid for general algebraic maps.}

In fact, since degenerations are a powerful tool for studying both quot schemes and the dimensions of spaces of sections arising in strange duality, my ultimate goal is to unite Problems 1 and 2 within one framework, as in the case of the wTQFT for curves. Since the cobordism ring has been shown in [EGL01] to be useful for (a), such a joint theory could involve computing cobordism classes of quot schemes on surfaces and proving that these classes satisfy degeneration relations.

**Main problem.** Develop a theory of degenerations of quot schemes on del Pezzo surfaces with an interpretation involving sections of determinant line bundles. Use this theory to find power series that simultaneously perform (a) and (c), thereby proving strange duality.

Moving away from del Pezzo surfaces, the methods in [BGJ16] could still be used to study strange duality. In particular, if $S$ is $K$-trivial, the dimensions in (a) are known ([MO08b]) but the quot schemes arising from orthogonal classes $\sigma, \rho$ will never be finite, even though their expected dimension is 0. Nevertheless, the kernels $E_i$ and quotients $F_i$ still produce subspaces of $\Gamma_{\sigma}$ and $\Gamma_{\rho}$.

**Problem 4.** In the case when Quot($V, \rho$) is infinite, find a way to select kernels $E_i$ and quotients $F_i$ that yield bases of $\Gamma_{\sigma}$ and $\Gamma_{\rho}$ on which SD$_{\sigma, \rho}$ is non-degenerate.

An appropriate theory of virtual fundamental classes of quot schemes, which is already necessary in the case of the wTQFT for curves and will likely be necessary for studying degenerations on del Pezzo surfaces, could also resolve the problem of defective quot schemes by picking out finitely many quotients and kernels.
**Derived category.** The determinant line bundles involved in strange duality are interesting from the point of view of the derived category and the minimal model program. The orthogonal class $\sigma$ defines a Bridgeland stability condition on the derived category of the surface $S$, and the associated moduli space $M_\sigma(\rho)$ of $\sigma$-semistable complexes with Chern character $\rho$ is birational to the moduli space of Gieseker-semistable sheaves. In [BM14], Bayer and Macrì prove that the determinant line bundle induced by $\sigma$ is nef (and often ample) on $M_\sigma(\rho)$. Varying the class $\sigma$ thus corresponds to running a minimal model program on the moduli space of sheaves with Chern character $\rho$ (see for example [ABCH13], [Tod14], [BM14], [BMW13]). This is a hint that $M_\sigma(\rho)$ should be the right space on which to study the determinant line bundle induced by $\sigma$.

While working on [BGJ16], we found evidence that moduli of complexes can be used to extend the quot scheme approach to more general pairs of orthogonal classes. In the special case when $\rho$ and $\sigma$ each have rank 1, the moduli spaces in strange duality are Hilbert schemes, and the determinant line bundles are pullbacks of $\mathcal{O}(1)$ under maps from these Hilbert schemes to complementary Grassmannians. The usual quot scheme method cannot succeed because ideal sheaves are not locally free. Nevertheless, we produce a vector bundle $V$ that has the correct number of non-surjective maps to ideal sheaves; viewed in an appropriate tilt of the category of coherent sheaves, these maps are surjections and the kernels are derived duals of ideal sheaves. Thus the quot scheme $\text{Quot}(V, \rho)$ in this tilted category is the right object for proving strange duality.

**Problem 5.** Use quot schemes for abelian subcategories of the derived category to extend the outline for proving strange duality to other pairs of orthogonal classes $\rho, \sigma$.

**References**


