1 Rational and Jordan Canonical Forms

For the rational form (over any field), the cyclic decomposition theorem guarantees the existence of non-zero vectors $\alpha_1, \ldots, \alpha_r$ with respective $T$-annihilators $p_1, \ldots, p_r$ such that

$$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$$

and

$$p_k | p_{k-1}, \quad k = 2, \ldots, r,$$

with $p_1 = p$ the minimal polynomial of $T$, $p_1 \cdots p_r = f$ the characteristic polynomial, and $\deg p_i = \dim Z(\alpha_i; T)$. Then in the basis

$$\{\alpha_1, T\alpha_1, \ldots, T^{\deg p_i - 1} \alpha_1, \alpha_2, T\alpha_2, \ldots\},$$

$T$ is composed of blocks

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & 0 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -a_{k-1}
\end{pmatrix},$$

where $p_i = x^k + a_{k-1} x^{k-1} + \cdots + a_0$.

The Jordan form (over an algebraically closed field like $\mathbb{C}$) is obtained by combining the primary decomposition theorem with the rational form, as follows. Given $A$ with characteristic polynomial $f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ and minimal polynomial $p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$, with $1 \leq r_i \leq d_i$ for each $i$, we use the primary decomposition theorem to write

$$V = W_1 \oplus \cdots \oplus W_k,$$

where $W_i$ is the null space of $(T - c_i I)^{r_i}$, and $W_i$ has dimension $d_i$. We choose a basis for $W_i$ corresponding to the cyclic decomposition of the nilpotent operator

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\( N_i := T - c_i I \) on \( W_i \). Since \( N_i \) has minimal polynomial \( x^{r_i} \) and \( T = N_i + c_i I \), the matrix for \( T \) in the chosen basis is composed of blocks of the form

\[
\begin{pmatrix}
c_i & 0 & \cdots & 0 & 0 \\
1 & c_i & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & c_i
\end{pmatrix},
\]

decreasing in size, with the first of size \( r_i \) and the sum of the sizes equal to \( d_i \). Putting these result together for all the \( W_i \) gives the Jordan form.

## 2 Special Matrices

Let \( V \) be a finite-dimensional inner product space.

- For every self-adjoint linear operator \( T \) on \( V \), there is an orthonormal basis of \( V \) consisting of characteristic vectors of \( T \).
- It follows that Hermitian matrices are diagonalizable by unitary matrices, and real symmetric matrices are diagonalizable by real orthogonal matrices.
- Even stronger: if \( T \) is a normal operator on \( V \), there is an orthonormal basis of \( V \) consisting of characteristic vectors of \( T \). Hence every normal matrix is diagonalizable by a unitary matrix.