Test #1

Solving a Linear System (10 points)

Consider the homogeneous linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 - x_3 &= 0 \end{aligned}$$

(a) Write the linear system as a matrix equation $A\vec{x} = \vec{b}$. **(2 points)**

(b) Write the linear system as a vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$. **(2 points)**

(c) Compute the reduced echelon form of the augmented matrix of the linear system. **(3 points)**

(d) Compute the solution set of the linear system and write the solution set as the span of a vector. **(3 points)**

Inventions (10 points)

(a) Invent a linear system of two equations in two variables that has infinitely many solutions. **(2 points)**

(b) Invent a vector $\vec{v}$ with no zero entries that is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. **(2 points)**

(c) Invent a 2×4 matrix A whose columns do not span $\mathbb{R}^2$. **(2 points)**

(d) Invent a vector $\vec{b}$ in $\mathbb{R}^3$ so that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$ has no solution. **(2 points)**

(e) Invent a matrix A so that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a solution of $A\vec{x} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. **(2 points)**

Solving Another Linear System (10 points)

Consider the linear system $\begin{bmatrix} 0 & 0 \\ -6 & -12 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -12 \\ 2 \\ 2 \end{bmatrix}$.

(a) Compute the reduced echelon form of the augmented matrix of the linear system. **(3 points)**

(b) Write the solution set of the system in parametric vector form. **(3 points)**

(c) Sketch the solution set in the plane $\mathbb{R}^2$. **(2 points)**

(d) Is this solution set the span of a vector? Why or why not? **(2 points)**

Span (10 points)

(a) Sketch the set $\text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$ in the plane $\mathbb{R}^2$. **(2 points)**

(b) Sketch the set $\text{Span}\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ in $\mathbb{R}^2$. **(2 points)**

(c) What can you deduce about vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ if $\text{Span}\{\vec{v}, \vec{w}\}$ is just one point? **(2 points)**

(d) What can you deduce about vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ if $\text{Span}\{\vec{v}, \vec{w}\}$ is a line? **(2 points)**

(e) Suppose you know that $\text{Span}\{\vec{v}, \vec{w}\}$ is a plane in $\mathbb{R}^3$. Suppose $\vec{u}$ in $\mathbb{R}^3$ is not on that plane. What is $\text{Span}\{\vec{v}, \vec{w}, \vec{u}\}$? **(2 points)**

Echelon Form (10 points)

Consider the matrix $A = \begin{bmatrix} 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Describe a row operation in words that would put A into reduced echelon form. **(2 points)**

(b) Which columns of A are the pivot columns? **(2 point)**

(c) In the equation $A\vec{x} = \vec{b}$, which are the free variables? **(2 point)**

(d) What can you say about $\vec{b}$ in $\mathbb{R}^3$ if you know that $A\vec{x} = \vec{b}$ has no solution? **(1 points)**

(e) Describe the set of all $\vec{b}$ in $\mathbb{R}^3$ for which $A\vec{x} = \vec{b}$ has exactly one solution. **(1 points)**

(f) Suppose that a matrix B can be row reduced to get the echelon matrix A. What can you deduce about the first column of B? What can you deduce about the last row of B? **(2 points)**

Test #2

Inventions (10 points)

(a) Invent a 3×3 matrix A that is not invertible. **(2 points)**

(b) Invent 2×2 matrices A and B such that $AB \neq BA$. **(2 points)**

(c) Invent a 2×3 matrix A with no zero entries such that the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ is not onto. **(2 points)**

(d) Invent a 2×2 matrix A such that $A \neq I_2$ and $A^2 = I_2$. **(2 points)**

(e) Invent a set of linearly independent vectors in $\mathbb{R}^3$ that contains as many vectors as possible. **(2 points)**

Inverse Matrix (10 points)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$.

(a) Let B be the inverse of A. Compute B using row reduction on $\begin{bmatrix} A & I_3 \end{bmatrix}$. **(5 points)**

(b) Show B is the inverse of A by computing BA. **(2 points)**

(c) Use B to solve $A\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. **(3 points)**

Geometry (10 points)

Let $A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$.

(a) In one copy of $\mathbb{R}^2$, draw Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and Span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. In another copy of $\mathbb{R}^2$, draw what you get when T acts on all points of those two lines. **(4 points)**

(b) In one copy of $\mathbb{R}^2$, draw the square with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In another copy of $\mathbb{R}^2$, draw what you get when T acts on all points of that square. **(4 points)**

(c) What geometric shape would you get if T acts on all points of a line that does not contain the origin? **(1 point)**

(d) What geometric shape would you get if T acts on all points of a parallelogram that is far from the origin? **(1 point)**

One-to-one (10 points)

Let $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ -1 & 2 \end{bmatrix}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$.

(a) Write down a linear dependence relation among the columns of A. **(2 points)**

(b) Based on your answer in (a), is T one-to-one? **(1 point)**

(c) Invent two distinct vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ such that $T(\vec{x}) = T(\vec{y})$. *Hint: try to use your answer in (a).* **(2 points)**

(d) Compute the echelon form of A. **(3 points)**

(e) Look at the pivots in your answer for (d). Is T one-to-one? Explain. **(2 points)**

Linear Transformations (10 points)

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(\vec{e}_1) = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

(a) Compute each of the following vectors if you can, or state that not enough information is given: **(3 points)**

(i) $T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$

(ii) $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

(iii) $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

Now suppose you also know that $T(\vec{e}_3) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$.

(b) Compute all vectors you couldn't compute in (a). **(3 points)**

(c) Compute a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^3$. **(4 points)**

Test #3

Inventions (10 points)

(a) Invent a matrix A such that $\det A = -4$ and $\det(3A) = -36$. **(2 points)**

(b) Invent a square matrix A such that $\operatorname{rank}(A) = 2$ and $\det A = 0$. **(2 points)**

(c) Invent a vector space V whose objects are matrices such that $\dim V = 100$. **(2 points)**

(d) Invent a 2×2 matrix A such that $A^T \neq A$ and $\operatorname{Col} A = \operatorname{Row} A$. **(2 points)**

(e) Invent a basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ of $\mathbb{P}_2$ such that $[t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. **(2 points)**

Subspaces Associated to a Matrix (10 points)

Let $A = \begin{bmatrix} 1 & 6 & -3 \\ 0 & 0 & 0 \\ -2 & -12 & 6 \end{bmatrix}$.

(a) Compute the reduced echelon form of A. **(2 points)**

(b) Compute a basis for $\operatorname{Nul} A$. What is $\dim(\operatorname{Nul} A)$? **(4 points)**

(c) Write down a basis for $\operatorname{Col} A$. What is $\dim(\operatorname{Col} A)$? **(2 point)**

(d) Write down a basis for $\operatorname{Row} A$. What is $\dim(\operatorname{Row} A)$? **(2 point)**

Determinant (10 points)

(a) Let H be the set of all matrices A in $M_{2\times 2}$ such that $\det A = 0$. Answer each of the following questions. You do not need to justify your answers. **(4 points)**

  (i) Is $\vec{0}$ in H?

  (ii) Is H closed under vector addition?

  (iii) Is H closed under scalar multiplication?

  (iv) Is H a subspace of $M_{2\times 2}$?

(b) Is the set of polynomials $\{1 + 2t, 3t + 3t^2, 4 - t + 7t^2\}$ a basis for $\mathbb{P}_2$? Use a coordinate mapping to reduce the problem to computing the determinant of a 3×3 matrix. Show your work! **(6 points)**

Change of Basis (10 points)

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

(a) Compute the $\mathcal{B}$-coordinates of each of the following vectors in $\mathbb{R}^2$: **(3 points)**

(i) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) Write down the matrix $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ that converts $\mathcal{B}$-coordinates to standard coordinates. Multiply $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ by one of your answers in (a) to check that you get back the original vector. **(3 points)**

(c) Compute the matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ that converts standard coordinates to $\mathcal{B}$-coordinates. Compute $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. You should get your answers in (a). **(4 points)**

Coordinates and Linear Transformations (10 points)

Let $\mathbb{P}_2 \xrightarrow{T} \mathbb{R}^2$ be the linear transformation defined by $T(\vec{p}) = \begin{bmatrix} \vec{p}(-1) \\ \vec{p}(2) \end{bmatrix}$.

(a) Compute $T(1)$, $T(t)$, and $T(t^2)$. **(2 points)**

Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Using the coordinate mapping $\mathbb{P}_2 \xrightarrow{[\ ]_{\mathcal{E}}} \mathbb{R}^3$, we can view T as a linear transformation $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2$ defined by $S(\vec{x}) = A\vec{x}$.

(b) Write down the matrix A. *Hint: the columns of A are your answers in (a).* **(2 points)**

(c) Use the matrix A to check that $S([t^2]_{\mathcal{E}}) = T(t^2)$. **(1 point)**

(d) Compute a basis for $\operatorname{Nul} A$. **(4 points)**

(e) Use your answer in (d) to write down a basis for the kernel of T. **(1 point)**

Test #4

Inventions (10 points)

(a) Invent a matrix A and a nonzero vector $\vec{x}$ that is not an eigenvector of A. **(2 points)**

(b) Invent a 2×2 matrix A such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a 3-eigenvector of A. **(2 points)**

(c) Invent a matrix A with all entries nonzero that has 0 as an eigenvalue. **(2 points)**

(d) Invent a matrix A that has a two-dimensional eigenspace. **(2 points)**

(e) Invent a 2×2 matrix A with real eigenvalues that is not diagonalizable. **(2 points)**

Definition of Eigenvectors (10 points)

(a) Write down the equation that defines what it means for $\vec{x}$ to be a λ-eigenvector of a matrix A. **(2 points)**

(b) Show that $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 7 \\ 0 & -1 \end{bmatrix}$ and compute its eigenvalue λ. **(2 points)**

(c) Suppose $\vec{x}$ is a 2-eigenvector of A and a 5-eigenvector of B. Show that $\vec{x}$ is an eigenvector of AB and compute its eigenvalue λ. **(2 points)**

(d) Compute a basis for the 6-eigenspace of the matrix $A = \begin{bmatrix} 3 & 0 & 2 \\ -6 & 6 & 4 \\ -3 & 0 & 8 \end{bmatrix}$. **(4 points)**

Diagonalization (10 points)

Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Compute the eigenvalues λ_1, λ_2 of A. **(2 points)**

(b) Compute independent eigenvectors $\vec{v}_1$, $\vec{v}_2$ of A. **(4 points)**

(c) Write down the diagonalization $A = PDP^{-1}$, where the columns of P are eigenvectors and D is diagonal. **(2 points)**

(d) Use your answer in (c) to write down a nice formula for A^{2015}. **(2 points)**

Matrix for the Derivative (10 points)

Let $\mathbb{P}_3 \xrightarrow{T} \mathbb{P}_2$ be the linear transformation that takes the derivative. For example, $T(5 + t^2 - 3t^3) = 2t - 9t^2$. Let $\mathcal{E} = \{1, t, t^2, t^3\}$ and $\mathcal{E}' = \{1, t, t^2\}$ denote the standard bases for $\mathbb{P}_3$ and $\mathbb{P}_2$.

(a) Write down the $\mathcal{E}$-coordinates of $5 + t^2 - 3t^3$ and the $\mathcal{E}'$-coordinates of $2t - 9t^2$. **(2 points)**

(b) Compute the $\mathcal{E}'$-coordinates of $T(1)$, $T(t)$, $T(t^2)$, and $T(t^3)$. **(4 points)**

(c) Write down the 3×4 matrix for T relative to the bases $\mathcal{E}$ and $\mathcal{E}'$. **(2 points)**

(d) Check your matrix is correct by multiplying it by the $\mathcal{E}$-coordinates of $5 + t^2 - 3t^3$ you found in (a). You should get the $\mathcal{E}'$-coordinates of $2t - 9t^2$. **(2 points)**

Discrete Dynamical System (10 points)

Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{1}{3}$ and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be the eigenvector basis for $\mathbb{R}^2$.

(a) Compute the change of basis matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$. **(3 points)**

(b) Let $\vec{x}_0 = \begin{bmatrix} 30 \\ 50 \end{bmatrix}$. Compute c_1 and c_2 such that $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$. *(Hint:* $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \vec{x}_0$.*)* **(2 points)**

(c) Use your answer in (b) to write down a nice formula for $A^k\vec{x}_0$. **(2 points)**

(d) What vector does $A^k\vec{x}_0$ converge to as k gets large? **(1 point)**

(e) Invent a vector $\vec{y}_0$ such that $A^k\vec{y}_0$ converges to $\vec{0}$ as k gets large. **(1 point)**

(f) Invent a vector $\vec{z}_0$ such that $A^k\vec{z}_0$ converges to $\vec{z}_0$ as k gets large. **(1 point)**

Test #5

Inventions (10 points)

(a) Invent a unit vector in $\mathbb{R}^4$ with no zero entries. **(2 points)**

(b) Invent a vector with no zero entries that is orthogonal to $\begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}$. **(2 points)**

(c) Invent vectors $\vec{x}_1, \vec{x}_2$ in $\mathbb{R}^2$ such that the orthogonal complement of $\text{Span}\{\vec{x}_1, \vec{x}_2\}$ is a line. **(2 points)**

(d) Invent a symmetric 2×2 matrix of rank 1. **(2 points)**

(e) Invent a matrix that is diagonalizable but not orthogonally diagonalizable. **(2 points)**

Orthogonal Projection (10 points)

Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

(a) Compute an orthogonal basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ of W. **(3 points)**

(b) Let $\vec{y} = \begin{bmatrix} -9 \\ 2 \\ 4 \end{bmatrix}$. Compute $\text{proj}_W \vec{y}$. **(4 points)**

(c) What is the closest point in W to $\vec{y}$? **(1 point)**

(d) Compute the distance from $\vec{y}$ to W. **(2 points)**

Least Squares Approximation (10 points)

Consider the equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$.

(a) Determine whether $A\vec{x} = \vec{b}$ has a solution. **(2 points)**

(b) Compute the unique solution $\vec{x}_0$ of $A^T A\vec{x} = A^T\vec{b}$. **(5 points)**

(c) Compute $\|A\vec{x}_0 - \vec{b}\|$. **(2 points)**

(d) What can you say about $\vec{x}_0$ to relate it to the original equation $A\vec{x} = \vec{b}$? **(1 point)**

Orthogonal Diagonalization (10 points)

Let $A = \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$.

(a) Why can you immediately tell that A is diagonalizable? **(1 point)**

(b) Compute the eigenvalues of A. **(3 points)**

(c) Compute a basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ consisting of orthonormal eigenvectors of A. **(4 points)**

(d) Write down the diagonalization $A = PDP^T$, where the columns of P are orthonormal eigenvectors and D is diagonal. **(2 points)**

Singular Value Decomposition (10 points)

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and assume that an orthogonal diagonalization of A^TA is given by

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}.$$

(a) Compute the singular value decomposition $A = U\Sigma V^T$. *(Hint: V and Σ can easily be obtained from the given information!)* **(7 points)**

(b) What is the rank r of A? **(1 point)**

(c) Compute the best rank 1 approximation of A. **(2 points)**