Test #5 Solutions

1 Inventions (10 points)

(a) Invent a unit vector in $\mathbb{R}^4$ with no zero entries. **(2 points)**

Possible solution:

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

(b) Invent a vector with no zero entries that is orthogonal to $\begin{bmatrix}3\\7\\-2\end{bmatrix}$. **(2 points)**

Possible solution:

$$\begin{bmatrix}1\\1\\5\end{bmatrix}$$

(c) Invent vectors $\vec{x}_1, \vec{x}_2$ in $\mathbb{R}^2$ such that the orthogonal complement of $\text{Span}\{\vec{x}_1, \vec{x}_2\}$ is a line. **(2 points)**

Possible solution:

$$\vec{x}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix}2\\0\end{bmatrix}$$

(d) Invent a symmetric 2×2 matrix of rank 1. **(2 points)**

Possible solution:

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

(e) Invent a matrix that is diagonalizable but not orthogonally diagonalizable. **(2 points)**

Possible solution:

$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$$

2 Orthogonal Projection (10 points)

Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

(a) Compute an orthogonal basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ of W. **(3 points)**

Possible solution:

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{u}_2 = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} - \frac{18+1+5}{4+1+1} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

(b) Let $\vec{y} = \begin{bmatrix} -9 \\ 2 \\ 4 \end{bmatrix}$. Compute $\text{proj}_W \vec{y}$. **(4 points)**

Solution:

$$\text{proj}_W \vec{y} = \frac{-18+2+4}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{-9-6+4}{1+9+1} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

(c) What is the closest point in W to $\vec{y}$? **(1 point)**

Solution:

$$\begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

(d) Compute the distance from $\vec{y}$ to W. **(2 points)**

Solution:

$$\|\vec{y} - \text{proj}_W \vec{y}\| = \left\| \begin{bmatrix} -9 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix} \right\| = \sqrt{16+1+49} = \sqrt{66}$$

3 Least Squares Approximation (10 points)

Consider the equation $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$.

(a) Determine whether $A\vec{x} = \vec{b}$ has a solution. **(2 points)**

Solution:

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -11 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

Since the last row is all zeros except a nonzero number in the augmented column, the system has no solution.

(b) Compute the unique solution $\vec{x}_0$ of $A^T A\vec{x} = A^T\vec{b}$. **(5 points)**

Solution:

$$A^T A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}, \qquad A^T\vec{b} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$\vec{x}_0 = (A^T A)^{-1} A^T \vec{b} = \frac{1}{11} \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 11 \\ -11 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) Compute $\|A\vec{x}_0 - \vec{b}\|$. **(2 points)**

Solution:

$$A\vec{x}_0 - \vec{b} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{9+1+1} = \sqrt{11}$$

(d) What can you say about $\vec{x}_0$ to relate it to the original equation $A\vec{x} = \vec{b}$? **(1 point)**

Solution: $\vec{x}_0$ is the vector such that $A\vec{x}_0$ is the best approximation to $\vec{b}$ in $\operatorname{Col} A$.

4 Orthogonal Diagonalization (10 points)

Let $A = \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$.

(a) Why can you immediately tell that A is diagonalizable? **(1 point)**

Solution: A is symmetric.

(b) Compute the eigenvalues of A. **(3 points)**

Solution:

$$\begin{vmatrix} 4-\lambda & -2 \\ -2 & 7-\lambda \end{vmatrix} = \lambda^2 - 11\lambda + 24 = (\lambda - 8)(\lambda - 3),$$

so $\lambda_1 = 8$, $\lambda_2 = 3$.

(c) Compute a basis $\{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ consisting of orthonormal eigenvectors of A. **(4 points)**

Solution:

$$A - 8I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \qquad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(d) Write down the diagonalization $A = PDP^T$, where the columns of P are orthonormal eigenvectors and D is diagonal. **(2 points)**

Solution:

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

5 Singular Value Decomposition (10 points)

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ and assume that an orthogonal diagonalization of $A^T A$ is given by $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$.

(a) Compute the singular value decomposition $A = U\Sigma V^T$. *(Hint: V and Σ can easily be obtained from the given information!)* **(7 points)**

$$V = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \frac{1}{\sqrt{5}} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$$

(b) What is the rank r of A? **(1 point)**

Solution: $r = 2$

(c) Compute the best rank 1 approximation of A. **(2 points)**

Solution:

$$\sigma_1 \vec{u}_1 \vec{v}_1^T = \sqrt{10} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \frac{1}{\sqrt{5}} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$