Test #4 Study Guide

November 9, 2015

Format

- Time and date: Tuesday, November 17, 12:55-1:45 PM in LCB 225.
- Review session: Thursday, November 12, 12:55-1:45 PM in LCB 225.
- Possible review session: Monday, November 16, 4:45-6:15 PM in JWB 333, if I can make it (jury duty may interfere!).
- If you have questions you may want to meet with me this week!
- Sections covered: 4.7, 4.9, 5.1, 5.2, 5.3, 5.4, 5.6.
- Study Quizzes 6 and 7, the recommended problems, and especially the concepts I've emphasized in lecture.
- One page of inventions and four pages of multiple-part questions. 10 points per page; 50 points total. Worth 10% of the final grade.

Key Concepts and Skills

Eigenvectors and Eigenvalues

- Know the defining equation $A\vec{x} = \lambda\vec{x}$ for λ-eigenvectors of a square matrix A. We call λ the eigenvalue of $\vec{x}$ and demand that $\vec{x} \neq 0$, though $\lambda = 0$ is fine.
- Understand the geometric interpretation of a λ-eigenvector $\vec{x}$ of a matrix A: multiplication by A stretches $\vec{x}$ by the scalar λ.
- Be able to check whether a vector $\vec{x}$ is an eigenvector for a matrix A by computing $A\vec{x}$ and determining whether the result is a scalar times $\vec{x}$. If so, then that scalar is the eigenvalue of $\vec{x}$.
- Be able to compute the eigenvalues $\lambda_1, \dots, \lambda_k$ of an $n \times n$ matrix A and their algebraic multiplicities $n_1, \dots, n_k$ by factoring the characteristic polynomial $\det(A - \lambda I)$. Interpret n_i as the number of independent λ_i-eigenvectors you are hoping to compute. Note that $n_1 + \cdots + n_k = n$ since the characteristic polynomial has degree n.

- Understand why the eigenvalues of an upper-triangular or lower-triangular matrix are the entries on the diagonal.

- The λ_i-eigenspace of A is the subspace of all λ_i-eigenvectors of A together with $\vec{0}$. Understand why the λ_i-eigenspace of A is the same as $\mathrm{Nul}(A - \lambda_i I)$ and be able to compute a basis for the λ_i-eigenspace of A using parametric vector form. The dimension of the λ_i-eigenspace of A is denoted m_i and is called the geometric multiplicity of the eigenvalue λ_i. The geometric multiplicity m_i is the number of independent λ_i eigenvalues you were actually able to compute, and satisfies $1 \leq m_i \leq n_i$ when λ_i is an eigenvalue of A.

- A is diagonalizable if A is similar to a diagonal matrix D. A is diagonalizable exactly when there is a basis of $\mathbb{R}^n$ consisting of eigenvectors for A, in which case taking P to be the matrix whose columns are those eigenvectors and D to be the diagonal matrix with the eigenvalues on the diagonal gives you $A = PDP^{-1}$.

- Be able to run the diagonalization algorithm for an $n \times n$ matrix A:
  (a) Compute the eigenvalues $\lambda_1, \ldots, \lambda_k$ and their algebraic multiplicities $n_1, \ldots, n_k$.
  (b) Compute bases for the λ_i-eigenspaces and the geometric multiplicities $m_1, \ldots, m_k$.
  (c) If $m_i < n_i$ for some i, then A is not diagonalizable.
  (d) If $m_i = n_i$ for all i, then A is diagonalizable. Collecting all your bases in (b) yields a basis for $\mathbb{R}^n$. Using those basis vectors as the columns of a matrix P and placing the corresponding eigenvalues on the diagonal of D gives you $A = PDP^{-1}$.

- Be able to compute A^k when A has been diagonalized.

- Be able to invent matrices with specified eigenvalues, algebraic multiplicities, and geometric multiplicities.

- Be able to invent examples of matrices that are not diagonalizable.

- Understand why A is guaranteed to be diagonalizable when all the algebraic multiplicities are equal to 1.

Coordinates and Change of Basis

- Understand the definition of a basis $\mathcal{B}$ of a vector space V and the coordinate mapping $V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$.

- Be able to use coordinate mappings to interpret a linear transformation $V \xrightarrow{T} W$ as a transformation $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ and compute a matrix A such that $S(\vec{x}) = A\vec{x}$.

- Understand what it means for $n \times n$ matrices A and B to be similar: there is an invertible matrix P such that $A = PBP^{-1}$. Understand why this means that A and B are two incarnations of the same linear transformation, where A acts by multiplication on the standard coordinates, B acts by multiplication on $\mathcal{B}$-coordinates, and $P = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a change-of-coordinates matrix:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A\cdot} & \mathbb{R}^n \\ {\scriptstyle P^{-1}\cdot}\downarrow & & \uparrow{\scriptstyle P\cdot} \\ \mathbb{R}^n & \xrightarrow[B\cdot]{} & \mathbb{R}^n \end{array}$$

- Given a basis $\mathcal{B}$ for $\mathbb{R}^n$, be able to compute the change-of-coordinates matrices $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}^{-1}$. Given a second basis $\mathcal{C}$ for $\mathbb{R}^n$, be able to compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\,\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$.
- Given two bases $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$ for a vector space V, be able to compute the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix}$.

Markov Chains and Discrete Dynamical Systems

- Be able to create a stochastic matrix A to model a particular situation. Understand the meaning of acting on an initial state probability vector $\vec{x}_0$ by powers of A. Determine the long-term behavior of the system by computing the steady-state probability vector $\vec{q}$ (a 1-eigenvector of A). Show that this is the right long-term behavior by diagonalizing A and computing the limit $\lim_{k\to\infty} A^k$.
- Be able to create a matrix A that models a discrete dynamical system. Diagonalize A and focus on the largest eigenvector to understand the long-term behavior of the system.
- Understand what it means for the origin to be an attractor, repeller, or saddle point for a dynamical system $\vec{x}_{k+1} = A\vec{x}_k$. The eigenvalues are the rates of attraction or repulsion along the directions given by the eigenvectors.
- Fight against logging in the old-growth forests of the Pacific Northwest so that the spotted owls can survive!