Test #4 Solutions

1 Inventions (10 points)

(a) Invent a matrix A and a nonzero vector $\vec{x}$ that is not an eigenvector of A. **(2 points)**

Possible solution: $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) Invent a 2×2 matrix A such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a 3-eigenvector of A. **(2 points)**

Possible solution: $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) Invent a matrix A with all entries nonzero that has 0 as an eigenvalue. **(2 points)**

Possible solution: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(d) Invent a matrix A that has a two-dimensional eigenspace. **(2 points)**

Possible solution: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(e) Invent a 2×2 matrix A with real eigenvalues that is not diagonalizable. **(2 points)**

Possible solution: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

2 Definition of Eigenvectors (10 points)

(a) Write down the equation that defines what it means for $\vec{x}$ to be a λ-eigenvector of a matrix A. **(2 points)**

Solution: $A\vec{x} = \lambda\vec{x}$

(b) Show that $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 7 \\ 0 & -1 \end{bmatrix}$ and compute its eigenvalue λ. **(2 points)**

Solution: $\begin{bmatrix} 1 & 7 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix} = - \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, so $\lambda = -1$.

(c) Suppose $\vec{x}$ is a 2-eigenvector of A and a 5-eigenvector of B. Show that $\vec{x}$ is an eigenvector of AB and compute its eigenvalue λ. **(2 points)**

Solution: $(AB)\vec{x} = A(B\vec{x}) = A(5\vec{x}) = 10\vec{x}$, so $\lambda = 10$.

(d) Compute a basis for the 6-eigenspace of the matrix $A = \begin{bmatrix} 3 & 0 & 2 \\ -6 & 6 & 4 \\ -3 & 0 & 8 \end{bmatrix}$. **(4 points)**

Solution: $A - 6I = \begin{bmatrix} -3 & 0 & 2 \\ -6 & 0 & 4 \\ -3 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} -3 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The solutions are $\vec{x} = \begin{bmatrix} \frac{2}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$, so a basis for $\text{Nul}(A - 6I)$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\}$.

3 Diagonalization (10 points)

Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Compute the eigenvalues λ_1, λ_2 of A. **(2 points)**

Solution: $\det(A - \lambda I) = (3-\lambda)(2-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$, so $\lambda_1 = 4$ and $\lambda_2 = 1$.

(b) Compute independent eigenvectors $\vec{v}_1$, $\vec{v}_2$ of A. **(4 points)**

Solution: $A - 4I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, so $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$A - I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, so $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(c) Write down the diagonalization $A = PDP^{-1}$, where the columns of P are eigenvectors and D is diagonal. **(2 points)**

Solution: $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

(d) Use your answer in (c) to write down a nice formula for A^{2015}. **(2 points)**

Solution: $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}^{2015} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4^{2015} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

4 Matrix for the Derivative (10 points)

Let $\mathbb{P}_3 \xrightarrow{T} \mathbb{P}_2$ be the linear transformation that takes the derivative. For example, $T(5+t^2-3t^3) = 2t - 9t^2$. Let $\mathcal{E} = \{1, t, t^2, t^3\}$ and $\mathcal{E}' = \{1, t, t^2\}$ denote the standard bases for $\mathbb{P}_3$ and $\mathbb{P}_2$.

(a) Write down the $\mathcal{E}$-coordinates of $5 + t^2 - 3t^3$ and the $\mathcal{E}'$-coordinates of $2t - 9t^2$. **(2 points)**

Solution: $[5+t^2-3t^3]_{\mathcal{E}} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ -3 \end{bmatrix}$, $[2t-9t^2]_{\mathcal{E}'} = \begin{bmatrix} 0 \\ 2 \\ -9 \end{bmatrix}$.

(b) Compute the $\mathcal{E}'$-coordinates of $T(1)$, $T(t)$, $T(t^2)$, and $T(t^3)$. **(4 points)**

Solution: $T(1) = 0$, so $[T(1)]_{\mathcal{E}'} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $T(t) = 1$, so $[T(t)]_{\mathcal{E}'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$T(t^2) = 2t$, so $[T(t^2)]_{\mathcal{E}'} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. $T(t^3) = 3t^2$, so $[T(t^3)]_{\mathcal{E}'} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

(c) Write down the 3×4 matrix for T relative to the bases $\mathcal{E}$ and $\mathcal{E}'$. **(2 points)**

Solution: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(d) Check your matrix is correct by multiplying it by the $\mathcal{E}$-coordinates of $5 + t^2 - 3t^3$ you found in (a). You should get the $\mathcal{E}'$-coordinates of $2t - 9t^2$. **(2 points)**

Solution: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -9 \end{bmatrix}$

5 Discrete Dynamical System (10 points)

Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{1}{3}$ and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be the eigenvector basis for $\mathbb{R}^2$.

(a) Compute the change of basis matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$. **(3 points)**

Solution: $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

(b) Let $\vec{x}_0 = \begin{bmatrix} 30 \\ 50 \end{bmatrix}$. Compute c_1 and c_2 such that $\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2$. *(Hint:* $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \vec{x}_0$.*)* **(2 points)**

Solution: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 30 \\ 50 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 140 \\ 70 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

(c) Use your answer in (b) to write down a nice formula for $A^k \vec{x}_0$. **(2 points)**

Solution: $A^k \vec{x}_0 = 20 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 10 \frac{1}{3^k} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d) What vector does $A^k \vec{x}_0$ converge to as k gets large? **(1 point)**

Solution: $\begin{bmatrix} 40 \\ 20 \end{bmatrix}$

(e) Invent a vector $\vec{y}_0$ such that $A^k \vec{y}_0$ converges to $\vec{0}$ as k gets large. **(1 point)**

Possible solution: $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(f) Invent a vector $\vec{z}_0$ such that $A^k \vec{z}_0$ converges to $\vec{z}_0$ as k gets large. **(1 point)**

Possible solution: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$