Test #3 Study Guide

October 24, 2015

Format

- Time and date: Tuesday, October 27, 12:55-1:45 PM in LCB 225.
- Review session: Monday, September 28, 4:45-6:15 PM in JWB 333.
- If you have questions but cannot attend the review session: email me to make an appointment.
- Sections covered: 3.1, 3.2, 3.3, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6.
- Study Quizzes 4 and 5, the recommended problems, the worksheet on coordinates and linear transformations, and especially the concepts I've emphasized in lecture.
- One page of inventions and four pages of multiple-part questions. 10 points per page; 50 points total. Worth 10% of the final grade.

Key Concepts and Skills

Determinants

- Be able to compute the determinant of a square matrix using row reduction or cofactor expansions.
- Know key properties of the determinant: row swaps change the sign; linear in each row separately; subtracting a multiple of one row from another does not change the determinant; $\det AB = \det A \det B$, etc.
- Understand why $\det A \neq 0$ exactly when A is invertible.
- If $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ is a linear transformation defined by $T(\vec{x}) = A\vec{x}$ for a 2×2 matrix A and if S is a subset of $\mathbb{R}^2$, then

  $$\text{area of } T(S) = |\det A| \cdot \text{area of S.}$$

  Be able to use this formula and the version for $\mathbb{R}^3$ to compute areas and volumes of simple geometric shapes.

Vector spaces, subspaces, and linear transformations

- Understand the idea of a vector space V: V is a set of objects, called vectors, with addition of vectors and scalar multiplication of vectors satisfying natural properties. Vector spaces are the most general setting in which linear combinations make sense.
- Know the key examples of vectors spaces: $\mathbb{R}^n$, $\mathbb{P}_n$, $M_{m\times n}$.
- Understand the definition of a subspace H of a vector space V. The idea is that a subspace is a vector space H sitting inside a bigger vector space V. Know how to check whether a subset H is a subspace by checking three conditions: (a) Is $\vec{0}$ in H? (b) Is H closed under vector addition? (c) Is H closed under scalar multiplication. Also recall that subspaces and spans of vectors are the same thing!
- Know some examples and non-examples of subspaces in $\mathbb{R}^n$, $\mathbb{P}_n$, and $M_{m\times n}$.
- Understand the definition of a linear transformation $V \xrightarrow{T} W$: the function T must satisfy $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ and $T(c\vec{v}) = cT(\vec{v})$.
- For a linear transformation $V \xrightarrow{T} W$, understand the kernel of T (which is a subspace of V) and the range of T (which is a subspace of W).
- Know examples of linear transformations: multiplication by a matrix for $\mathbb{R}^n \to \mathbb{R}^m$; the derivative on $\mathbb{P}_n$; evaluation at some values of t on $\mathbb{P}_n$; coordinate mappings.

Bases and coordinates

- Understand linear independence, dependence, and dependence relations in $\mathbb{R}^n$, $\mathbb{P}_n$, and $M_{m\times n}$. Understand spans of vectors in these vector spaces.
- Understand the definition of a basis $\mathcal{B}$ of a vector space or subspace: $\mathcal{B}$ is an ordered list of vectors that are independent and span the vector space or subspace. A basis is a minimal set of vectors that spans a vector space or subspace.
- Know the standard basis $\mathcal{E}$ of $\mathbb{R}^n$, $\mathbb{P}_n$, and $M_{m\times n}$. Know how to construct other bases of these vector spaces. Understand why all bases of $\mathbb{R}^n$ are the columns of invertible $n \times n$ matrices. Know how to construct a basis of a subspace by inspection.
- The dimension of a vector space or subspace is defined to be the number of vectors in a basis for that vector space or subspace.
- A basis $\mathcal{B}$ of a vector space V, where $\dim V = n$, gives a coordinate mapping $V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$. Know how to compute $\mathcal{B}$-coordinates for vectors in V: if $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, then for any $\vec{v}$, there are unique scalars $c_1, \dots, c_n$ such that $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, and we define the $\mathcal{B}$-coordinates of $\vec{v}$ as the column of these scalars:

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

- Coordinate mappings are linear, one-to-one, and onto (thus they are invertible as functions). We call coordinate mappings "isomorphisms" because they give a perfect correspondence between the vectors in V and in $\mathbb{R}^n$ and between linear algebra concepts like linear independence, dependence, and span in V and in $\mathbb{R}^n$.

- Be able to use a coordinate mapping to determine whether sets of vectors in $\mathbb{P}_n$ and $M_{m\times n}$ are independent, whether they span the vector space, and whether they are a basis. The coordinate mapping lets you answer these questions in $\mathbb{R}^n$, where you have access to the powerful tool known as row reduction.

- Be able to use coordinate mappings to study linear transformations. In the case of a linear transformation $V \xrightarrow{T} \mathbb{R}^m$, use a coordinate mapping on V to view T as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, where $n = \dim V$ (for example, see the worksheet). In the case of $V \xrightarrow{T} W$, use a coordinate mapping on V and a coordinate mapping on W to view T as as linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, where $n = \dim V$ and $m = \dim W$ (example from lecture: the derivative). Then find the matrix for the transformation $\mathbb{R}^n \to \mathbb{R}^m$ and use that matrix to study the transformation (in particular, by computing the kernel of T and the range of T; see below).

Computation in $\mathbb{R}^n$

- To determine whether a list of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is independent, row reduce the matrix $A = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_p\end{bmatrix}$ and check whether every column has a pivot. (The vectors can never be independent if $p > n$.)

- To determine whether a list of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ spans $\mathbb{R}^n$, row reduce the matrix $A = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_p\end{bmatrix}$ and check whether every row has a pivot. (The vectors can never span $\mathbb{R}^n$ if $p < n$.)

- To determine whether a list of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $\mathbb{R}^n$, check whether the matrix $A = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_p\end{bmatrix}$ is invertible (for instance by computing $\det A$). (The vectors can never be a basis if $p \neq n$.)

- To check whether a list of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in any vector space V is independent, spans V, or is a basis for V, choose a basis $\mathcal{B}$ for V (taking the standard basis $\mathcal{E}$ is usually easiest) and use the coordinate mapping $V \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$ to translate the question into a question about vectors in $\mathbb{R}^n$, which you can solve as described above.

- When $V = \mathbb{R}^n$, $\mathcal{B}$-coordinates give us a new way of viewing vectors in $\mathbb{R}^n$ as column vectors. Understand why the coordinate mapping $\mathbb{R}^n \xrightarrow{[\ ]_{\mathcal{B}}} \mathbb{R}^n$ is multiplication by the matrix $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$, which is the inverse of the matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ whose columns are the vectors in the basis $\mathcal{B}$. Be able to use this matrix to compute $\mathcal{B}$-coordinates of vectors in $\mathbb{R}^n$.

Subspaces associated to an $m \times n$ matrix A

- $\operatorname{Nul} A$, the set of all solutions of $A\vec{x} = \vec{0}$, is a subspace of $\mathbb{R}^n$.
- $\operatorname{Col} A$, the span of the columns of A, is a subspace of $\mathbb{R}^m$.
- $\operatorname{Row} A$, the span of the rows of A viewed as columns (by transposing them), is a subspace of $\mathbb{R}^n$.
- Understand why $\operatorname{Nul} A$, $\operatorname{Col} A$, and $\operatorname{Row} A$ are subspaces.
- Know how to find bases for $\operatorname{Nul} A$, $\operatorname{Col} A$, and $\operatorname{Row} A$ by row reducing A:
  - The vectors that appear in the parametric vector form of the set of solutions to $A\vec{x} = \vec{0}$ (you will need the reduced echelon form of A) are a basis for $\operatorname{Nul} A$. Thus $\dim(\operatorname{Nul} A) = \#$ of non-pivot columns in A.
  - The pivot columns (taken from A) are a basis for $\operatorname{Col} A$. Thus $\dim(\operatorname{Col} A) = \#$ of pivot columns in A.
  - The pivot rows (taken from the echelon form or reduced echelon form of A) are a basis for $\operatorname{Row} A$. Thus $\dim(\operatorname{Row} A) = \#$ of pivot rows in A.
- Know the definition $\operatorname{rank} A = \dim(\operatorname{Col} A) = \dim(\operatorname{Row} A) = \#$ of pivots.
- Understand how to get the useful formula $\operatorname{rank} A + \dim(\operatorname{Nul} A) = n$ from the obvious equation

$$(\#\text{ of pivot columns}) + (\#\text{ of non-pivot columns}) = \text{number of columns}.$$

- If $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is the linear transformation defined by $T(\vec{x}) = A\vec{x}$, understand why the kernel of T equals $\operatorname{Nul} A$ and the range of T equals $\operatorname{Col} A$.
- Know how to compute bases for the kernel and range of any linear transformation $V \xrightarrow{T} W$ by:
  - Use coordinate mappings to view T as $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$.
  - Compute the matrix A for S using the definition of T.
  - Compute bases for the kernel of S and the range of S using row reduction on A.
  - Convert the vectors in these bases back into vectors in V and W by undoing the coordinate mapping.

  For a detailed example of this procedure, see the worksheet on coordinates and linear transformations. In the worksheet, W is already $\mathbb{R}^2$, so you do not need a coordinate mapping for W.