Test #3 Solutions

1 Inventions (10 points)

(a) Invent a matrix A such that $\det A = -4$ and $\det(3A) = -36$. **(2 points)**

Possible solution: $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

(b) Invent a square matrix A such that $\text{rank}(A) = 2$ and $\det A = 0$. **(2 points)**

Possible solution: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) Invent a vector space V whose objects are matrices such that $\dim V = 100$. **(2 points)**

Possible solution: $V = M_{4\times 25}$

(d) Invent a 2×2 matrix A such that $A^T \neq A$ and $\text{Col}\, A = \text{Row}\, A$. **(2 points)**

Possible solution: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(e) Invent a basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ of $\mathbb{P}_2$ such that $[t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. **(2 points)**

Possible solution: $\mathcal{B} = \{t^2 - 1, 1, t\}$

2 Subspaces Associated to a Matrix (10 points)

Let $A = \begin{bmatrix} 1 & 6 & -3 \\ 0 & 0 & 0 \\ -2 & -12 & 6 \end{bmatrix}$.

(a) Compute the reduced echelon form of A. **(2 points)**

Solution: $\begin{bmatrix} 1 & 6 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) Compute a basis for Nul A. What is dim(Nul A)? **(4 points)**

Solution: The general solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} -6x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, so a basis for Nul A is $\left\{ \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$. Thus $\dim(\text{Nul}\, A) = 2$.

(c) Write down a basis for Col A. What is dim(Col A)? **(2 point)**

Solution: A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$, so $\dim(\text{Col}\, A) = 1$.

(d) Write down a basis for Row A. What is dim(Row A)? **(2 point)**

Solution: A basis for Row A is $\left\{ \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} \right\}$, so $\dim(\text{Row}\, A) = 1$.

3 Determinant (10 points)

(a) Let H be the set of all matrices A in $M_{2\times 2}$ such that $\det A = 0$. Answer each of the following questions. You do not need to justify your answers. **(4 points)**

(i) Is $\vec{0}$ in H? *Solution:* Yes.

(ii) Is H closed under vector addition? *Solution:* No.

(iii) Is H closed under scalar multiplication? *Solution:* Yes.

(iv) Is H a subspace of $M_{2\times 2}$? *Solution:* No.

(b) Is the set of polynomials $\{1+2t, 3t+3t^2, 4-t+7t^2\}$ a basis for $\mathbb{P}_2$? Use a coordinate mapping to reduce the problem to computing the determinant of a 3×3 matrix. Show your work! **(6 points)**

Solution:

$$[1+2t]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad [3t+3t^2]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \qquad [4-t+7t^2]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$$

Since

$$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 3 & 7 \end{vmatrix} = 1\begin{vmatrix} 3 & -1 \\ 3 & 7 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} = 24 + 24 = 48,$$

the matrix is invertible, so its columns are a basis of $\mathbb{R}^3$ and therefore the set of polynomials is a basis of $\mathbb{P}_2$.

4 Change of Basis (10 points)

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

(a) Compute the $\mathcal{B}$-coordinates of each of the following vectors in $\mathbb{R}^2$: **(3 points)**

(i) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ *Solution:* $\left[\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ *Solution:* $\left[\begin{bmatrix} 2 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ *Solution:* $\left[\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(b) Write down the matrix $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ that converts $\mathcal{B}$-coordinates to standard coordinates. Multiply $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ by one of your answers in (a) to check that you get back the original vector. **(3 points)**

Solution:

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

(c) Compute the matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ that converts standard coordinates to $\mathcal{B}$-coordinates. Compute $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. You should get your answers in (a). **(4 points)**

Solution:

$$\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}^{-1} = \frac{1}{-2} \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

5 Coordinates and Linear Transformations (10 points)

Let $\mathbb{P}_2 \xrightarrow{T} \mathbb{R}^2$ be the linear transformation defined by $T(\vec{p}) = \begin{bmatrix} \vec{p}(-1) \\ \vec{p}(2) \end{bmatrix}$.

(a) Compute $T(1)$, $T(t)$, and $T(t^2)$. **(2 points)**

Solution:

$$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad T(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad T(t^2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Using the coordinate mapping $\mathbb{P}_2 \xrightarrow{[\ ]_{\mathcal{E}}} \mathbb{R}^3$, we can view T as a linear transformation $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2$ defined by $S(\vec{x}) = A\vec{x}$.

(b) Write down the matrix A. *Hint: the columns of A are your answers in (a).* **(2 points)**

Solution:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

(c) Use the matrix A to check that $S([t^2]_{\mathcal{E}}) = T(t^2)$. **(1 point)**

Solution:

$$S([t^2]_{\mathcal{E}}) = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = T(t^2)$$

(d) Compute a basis for $\operatorname{Nul} A$. **(4 points)**

Solution:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

The general solution to $A\vec{x} = \vec{0}$ is

$$\vec{x} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix},$$

so a basis for $\operatorname{Nul} A$ is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(e) Use your answer in (d) to write down a basis for the kernel of T. **(1 point)**

Solution: A basis for the kernel of T is $\{-2 - t + t^2\}$.