Test #2 Study Guide

September 24, 2015

Format

- Time and date: Tuesday, September 29, 12:55-1:45 PM in LCB 225.
- Review session: Monday, September 28, 4:45-6:15 PM in JWB 333.
- Sections covered: 1.7, 1.8, 1.9, 2.1, 2.2, 2.3.
- Study Quizzes 2 and 3, the recommended problems, and especially the concepts I've emphasized in lecture.
- One page of inventions and four pages of multiple-part questions. 10 points per page; 50 points total. Worth 10% of the final grade.

Key Concepts and Skills

Linear dependence and span

- Understand intuitively and geometrically the definition of linear dependence, linear independence, and the span of a set of vectors.
- Be able to find linear dependence relations between simple sets of vectors by inspection.
- Use row-reduction on an $m \times n$ matrix A to determine whether $\{\text{columns of } A\}$ is a linearly independent set and whether $\text{Span}\{\text{columns of } A\} = \mathbb{R}^m$.

Linear transformations

- Understand the terminology associated to linear functions/transformations/maps $T\colon \mathbb{R}^n \to \mathbb{R}^m$: domain, codomain, range, image of a vector, one-to-one, onto.
- Know the two defining properties of a linear function T and the deduced property that $T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n)$, namely T takes linear combinations to linear combinations. Understand this geometrically to mean that T takes lines to lines, parallelograms to parallelograms, and more generally spans to spans.

- Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ can be thought of as $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$.

- Be able to sketch how geometric transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ (such as rotation, projection, and shear) act on vectors and geometric shapes like the square with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- Given $T\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ for an $m \times n$ matrix A, understand the many equivalent ways of expressing that T is one-to-one and onto:

$$\begin{aligned} T \text{ is one-to-one} &\iff \text{For every } \vec{b} \text{ in } \mathbb{R}^m,\ A\vec{x} = \vec{b} \text{ has at most one solution} \\ &\iff A\vec{x} = \vec{0} \text{ has only the trivial solution } \vec{x} = \vec{0} \\ &\iff \text{the columns of } A \text{ are linearly independent} \\ &\iff \text{every column of } A \text{ has a pivot position;} \end{aligned}$$

$$\begin{aligned} T \text{ is onto} &\iff A\vec{x} = \vec{b} \text{ has at least one solution for each } \vec{b} \text{ in } \mathbb{R}^m \\ &\iff \text{Span}\{\text{columns of } A\} = \mathbb{R}^m \\ &\iff \text{every row of } A \text{ has a pivot position.} \end{aligned}$$

  Recall the key intuition that the pivot columns of A are the largest set of independent columns among the columns of A. Be able to check all these properties for a transformation T defined by a particular matrix A.

Matrix operations

- Know how to compute AB when A is an $m \times n$ matrix and B is an $n \times p$ matrix. If $B = \begin{bmatrix} \vec{b}_1, \dots, \vec{b}_p \end{bmatrix}$, then $AB = [A\vec{b}_1, \dots, A\vec{b}_p]$, which shows that the columns of AB are linear combinations of the columns of A.

- Usually $AB \neq BA$, even when both products are defined!

- $(AB)^T = B^T A^T$. Know why the change in order of A and B is necessary by looking at the sizes of A^T and B^T.

Invertible matrices

- The inverse of a matrix is only possible if A is a square matrix!

- Recall the intuition that the inverse of a matrix is like the multiplicative inverse of a real number; in this sense non-invertible matrices are like the real number 0, which doesn't have a multiplicative inverse. If A is invertible, then $A\vec{x} = \vec{b}$ always has a unique solution, while if a square matrix A is non-invertible, then $A\vec{x} = \vec{b}$ has either no solution or infinitely many solutions. (This easy characterization into two cases only works when A is a square matrix!)

- Given an $n \times n$ matrix A, be able to check whether an $n \times n$ matrix B is the inverse of A: multiply AB or BA and see if you get I_n.

- The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ when $ad - bc \neq 0$.

- Understand how A^{-1} can be used to easily solve $A\vec{x} = \vec{b}$ (when A^{-1} exists).

- Be able to check using the definition of inverse that when both A and B are invertible $n \times n$ matrices, the inverse of AB is $B^{-1}A^{-1}$.

- Know how to use row reduction on the augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$ to compute A^{-1} if it exists. Understand what goes wrong if A is not invertible.

- Row reduction of large augmented matrices is hard work! Even if A is a square matrix, it is usually easier to solve a system $A\vec{x} = \vec{b}$ by row reducing the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ than by computing A^{-1} (if it exists). But if you happen to already know the inverse of A (for instance if A is 2×2), then use A^{-1}.

- Understand all the equivalences in the Invertible Matrix Theorem (Section 2.3) and be able to check them all for a particular $n \times n$ matrix A. These equivalent properties are essentially just the combination of the equivalences for one-to-one and onto.