Quiz #9 Solutions

(1) Let $\begin{bmatrix}0\\3\end{bmatrix}$, $\begin{bmatrix}1\\-1\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$ be data points in $\mathbb{R}^2$. The goal of this problem is to find the line $y = \beta_0 + \beta_1 x$ that best fits these data points.

(a) Write down the equation $X\vec{\beta} = \vec{y}$ that models the problem. **(1 point)** *Solution:*

$$\begin{bmatrix}1 & 0\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \begin{bmatrix}3\\-1\\1\end{bmatrix}$$

(b) Solve the equation $X^TX\vec{\beta} = X^T\vec{y}$ for $\vec{\beta}$. **(2 points)**

Solution:

$$X^TX = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}3 & 3\\3 & 5\end{bmatrix}, \qquad X^T\vec{y} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}3\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix},$$

so

$$\vec{\beta} = (X^TX)^{-1}X^T\vec{y} = \frac{1}{15-9}\begin{bmatrix}5 & -3\\-3 & 3\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}12\\-6\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}.$$

(c) Write down the equation of the line that best fits the data points. **(1 point)**

Solution: $y = 2 - x$

(d) Sketch the data points and your line of best fit. **(1 point)**

Description of solution: You should plot the points $\begin{bmatrix}0\\3\end{bmatrix}$, $\begin{bmatrix}1\\-1\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$ and the line with equation $y = 2 - x$, which has y-intercept 2 and x-intercept 2.

(2) Compute an orthogonal basis for the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$. **(2 points)**

Possible solution:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} - \frac{7}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11/2 \\ 1 \\ -5/2 \end{bmatrix}$$

(3) Bonus problem: Compute the equation of the form $y = \beta_0 \cos x + \beta_1 \sin x$ that best fits the data points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi/2 \\ 1 \end{bmatrix}, \begin{bmatrix} \pi \\ 1 \end{bmatrix}$. **(1 bonus point)**

Solution: In this case

$$X = \begin{bmatrix} \cos 0 & \sin 0 \\ \cos \pi/2 & \sin \pi/2 \\ \cos \pi & \sin \pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so the dream equation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$X^T X = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad X^T \vec{y} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so the solution to $X^T X \vec{\beta} = X^T \vec{y}$ is

$$\vec{\beta} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$$

Thus the best fit equation is $y = -\frac{1}{2} \cos x + \sin x$.