Quiz #8 Solutions

(1) Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$, where $\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$. Let $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$.

(a) Show that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal. **(1 point)**

Solution: $\vec{u}_1 \cdot \vec{u}_2 = -4 - 2 + 0 + 6 = 0$.

(b) Compute the closest point to $\vec{y}$ in W. **(3 points)**

Solution:

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 = \frac{30}{10}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26}\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

(c) Write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W. **(2 points)**

Solution:

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

(d) Compute the distance from $\vec{y}$ to W. **(1 point)**

Solution:

$$\|\vec{z}\| = \sqrt{4^2 + 4^2 + 4^2 + 4^2} = \sqrt{64} = 8.$$

(2) Bonus problem: Let W be the plane in $\mathbb{R}^3$ spanned by $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. **(1 bonus point)**

(a) Compute $\operatorname{proj}_W \vec{e}_1$, $\operatorname{proj}_W \vec{e}_2$, and $\operatorname{proj}_W \vec{e}_3$.

Solution: $\operatorname{proj}_W \vec{e}_1 = \vec{e}_1$ since $\vec{e}_1$ is already in W.

$$\operatorname{proj}_W \vec{e}_2 = \frac{0}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \qquad \operatorname{proj}_W \vec{e}_3 = \frac{0}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Write down the matrix A for the linear transformation proj_W.

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Compute a basis for the range of proj_W.

Solution: The range of proj_W is W (and also $\operatorname{Col} A$), which has basis $\{\vec{u}_1, \vec{u}_2\}$.

(d) Compute a basis for the kernel of proj_W.

Solution: The kernel of proj_W is $W^\perp$ and also $\operatorname{Nul} A$, which has basis $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.