Quiz #6 Solutions

(1) A dragon hunts down and eats either a horse or a linear algebra student each day. If the dragon eats a horse on a given day, it is 70% likely to eat a horse on the next day and 30% likely to eat a student. If the dragon eats a student on a given day, it is 90% likely to eat another student on the next day and only 10% likely to eat a horse instead.

(a) Write down a stochastic matrix P that models the situation. Label the columns and rows. **(1 point)**

Solution: $P = \begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix}$ (label the columns H and S and the rows H and S).

(b) Given that the dragon is 80% likely to eat a student today and only 20% likely to eat a horse, how likely is the dragon to eat a student tomorrow? **(1 point)**

Solution: $\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix} \begin{bmatrix} .2 \\ .8 \end{bmatrix} = \begin{bmatrix} .22 \\ .78 \end{bmatrix}$, so 78%.

(c) Compute the steady-state vector $\vec{q}$ by computing $\mathrm{Nul}(P - I)$. **(2 points)**

Solution:

$$P - I = \begin{bmatrix} -.3 & .1 \\ .3 & -.1 \end{bmatrix} \to \begin{bmatrix} -.3 & .1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix},$$

so the parametric vector form of a solution to $(P - I)\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$.
Thus $\mathrm{Nul}(P - I) = \mathrm{Span}\left\{ \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \right\}$. The probability vector in this span is

$$\vec{q} = \frac{1}{4/3} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} .25 \\ .75 \end{bmatrix}.$$

(d) Check that $P\vec{q} = \vec{q}$. **(1 point)**

Solution: $P\vec{q} = \frac{1}{4} \begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} .7 + .3 \\ .3 + 2.7 \end{bmatrix} = \vec{q}$

(e) Given that the dragon has eaten a student on November 3, 2015, how likely is the dragon to continue the tradition and eat a student on November 3, 2016? **(1 point)**

Solution: Exact answer: the second entry of $P^{365} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Rough answer: the second entry of $\vec{q}$, namely 75%.

(2) In $\mathbb{P}_2$, write down the change-of-coordinates matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ from the basis $\mathcal{B} = \{1+t, 2t-t^2, 3+4t-5t^2\}$ to the standard basis $\mathcal{E} = \{1, t, t^2\}$. **(1 point)**

Solution: $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 4 \\ 0 & -1 & -5 \end{bmatrix}$

(3) Bonus problem: Consider the stochastic matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. **(1 bonus point)**

(a) Find a formula for P^k.

Solution: $P^k = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } k \text{ is even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } k \text{ is odd} \end{cases}$

(b) Explain why no power of P has all positive entries. Thus P is not regular.

Solution: The formula in (a) shows that no power of P has all positive entries.

(c) Find the steady-state vector $\vec{q}$ for P. Check that $P\vec{q} = \vec{q}$.

Solution:

$$P - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

so $\text{Nul}(P - I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and the probability vector in this span is $\vec{q} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) Invent an initial state vector $\vec{x}_0$ such that the Markov chain $\vec{x}_k = P^k \vec{x}_0$ does not converge to $\vec{q}$ as $k \to \infty$. Explain why convergence fails.

Possible solution: $\vec{x}_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix}$. (Any probability vector $\vec{x}_0$ will work except $\vec{q}$.)