Quiz #5 Solutions

(1) For each of the following, determine whether H is a subspace. You do not need to justify your answer. **(3 points)**

(a) $H = \text{Span}\left\{\begin{bmatrix}7\\8\end{bmatrix}\right\}$ in $\mathbb{R}^2$.

Solution: Yes

Comment: Spans are subspaces and every subspace is a span!

(b) H is the set of all polynomials in $\mathbb{P}_3$ such that $\vec{p}(2) = 0$.

Solution: Yes

Comment: Check the three conditions for a subset to be a subspace.

(c) H is the set of all matrices in $M_{2\times 3}$ of the form $\begin{bmatrix}a & 0 & b\\0 & a & c\end{bmatrix}$.

Solution: Yes

Comment: Check that the 0-matrix fits the form and that the form is preserved under vector addition and scalar multiplication.

(2) Let $A = \begin{bmatrix}1 & 5 & 0 & -1 & 0\\0 & 0 & 1 & 1 & 0\end{bmatrix}$. List vectors that span $\text{Nul}\,A$. **(2 points)**

Solution: The free variables in the equation $A\vec{x} = \vec{0}$ are x_2, x_4, and x_5, and the general solution is

$$\vec{x} = \begin{bmatrix}-5x_2 + x_4\\x_2\\-x_4\\x_4\\x_5\end{bmatrix} = x_2\begin{bmatrix}-5\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix} + x_5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix},$$

so the vectors $\begin{bmatrix}-5\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ span $\text{Nul}\,A$.

Comment: Do a quick check to make sure the vectors you found satisfy $A\vec{x} = \vec{0}$.

(3) Determine whether $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. **(2 points)**

Possible solution: The vectors are not a basis because they are dependent:

$$\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Possible solution:

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 3 \\ 2 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

so the vectors are dependent and fail to span $\mathbb{R}^3$, hence they do not form a basis.

(4) Bonus problem: Determine whether $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}$. **(1 bonus point)**

Possible solution: The vectors are not a basis because they are dependent:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Possible solution: Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Computing $\mathcal{B}$-coordinates for the vectors in the proposed list, we get

$$\left[\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \left[\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \left[\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \left[\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

These coordinate vectors do not form a basis of $\mathbb{R}^4$ because

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 0$$

since two rows are the same. Thus the proposed vectors are not a basis of $M_{2\times 2}$.