Quiz #4 Solutions

(1) Compute $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. **(3 points)**

Possible solution:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Possible solution:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (45-48) - 2(36-42) + 3(32-35) \\ &= -3 + 12 - 9 \\ &= 0 \end{aligned}$$

Possible solution: The columns of the matrix are dependent since

$$2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix},$$

so the matrix is not invertible and hence its determinant is 0.

(2) Let A and B be 5×5 matrices, with $\det A = -2$ and $\det B = 3$. Use properties of determinants to compute each of the following: **(2 points)**

Solution:

(a) $\det BA = (\det B)(\det A) = 3(-2) = -6$

(b) $\det 2A = 2^5 \det A = -2^6 = -64$

(3) Compute the area of the parallelogram whose vertices are $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix}$. **(2 points)**

Solution: Let S be the unit square with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix}4 & -1\\3 & 2\end{bmatrix}$. Then $T(S)$ is the parallelogram we are interested in and its area is given by

$$\text{area of } T(S) = (\text{area of } S)\,|\det A| = 1 \cdot \left| \begin{vmatrix}4 & -1\\3 & 2\end{vmatrix} \right| = |8 - (-3)| = 11.$$

(4) Bonus problem: Let a and b be positive numbers. Compute the area of the region in $\mathbb{R}^2$ bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

(1 bonus point)

Solution: Let S be the region in $\mathbb{R}^2$ bounded by the unit circle, which has equation $u_1^2 + u_2^2 = 1$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{u}) = A\vec{u}$, where $A = \begin{bmatrix}a & 0\\0 & b\end{bmatrix}$. Then $T(S)$ is the ellipse we are interested in. (To see this, note that the image of a vector $\begin{bmatrix}u_1\\u_2\end{bmatrix}$ on the unit circle is the vector $T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = \begin{bmatrix}au_1\\bu_2\end{bmatrix}$, which satisfies the equation for the ellipse:

$$\frac{(au_1)^2}{a^2} + \frac{(bu_2)^2}{b^2} = u_1^2 + u_2^2 = 1.$$

Thus T maps the unit circle to the ellipse. Since T is linear, T must map the interior of the unit circle to the interior of the ellipse.) Now by the area formula,

$$\text{area of } T(S) = (\text{area of } S)\,|\det A| = \pi \cdot 1^2 \cdot \left| \begin{vmatrix}a & 0\\0 & b\end{vmatrix} \right| = \pi ab.$$