Final Exam Solutions

1 The Four Subspaces (10 points)

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$.

(a) Compute the rank r of A. **(1 point)**

Solution:

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad \text{has 2 pivots, so } r = 2.$$

(b) Use r to compute the dimensions of the four fundamental subspaces $N(A)$, $C(A)$, $C(A^T)$, $N(A^T)$. **(2 points)**

Solution: $\dim N(A) = 1$, $\dim C(A^T) = 2$, $\dim C(A) = 2$, $\dim N(A^T) = 0$.

(c) Which pairs of subspaces are orthogonal? **(1 point)**

Solution: $N(A)$ and $C(A^T)$; $C(A)$ and $N(A^T)$.

(d) Compute bases for the four fundamental subspaces of A. **(6 points)**

Solution: Basis for $N(A)$: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$. Basis for $C(A^T)$: $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Basis for $C(A)$: $\left\{ \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$. Basis for $N(A^T)$: $\{\}$

2 Inverse (10 points)

Let $A = \begin{bmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 0 & 4 & -10 \end{bmatrix}$.

(a) Compute the inverse A^{-1} of A using Gauss-Jordan elimination on $\begin{bmatrix} A & I \end{bmatrix}$ or the cofactor formula $\frac{1}{\det A}C^T$. **(8 points)**

Solution (Gauss-Jordan elimination):

$$\begin{bmatrix} -1 & 0 & 2 & 1 & 0 & 0 \\ 3 & -2 & 0 & 0 & 1 & 0 \\ 0 & 4 & -10 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & 6 & 3 & 1 & 0 \\ 0 & 4 & -10 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & 6 & 3 & 1 & 0 \\ 0 & 0 & 2 & 6 & 2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 0 & 0 & -5 & -2 & -1 \\ 0 & -2 & 0 & -15 & -5 & -3 \\ 0 & 0 & 2 & 6 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 & 2 & 1 \\ 0 & 1 & 0 & 15/2 & 5/2 & 3/2 \\ 0 & 0 & 1 & 3 & 1 & 1/2 \end{bmatrix}$$

(b) Check your inverse is correct by showing $A^{-1}A = I$. **(2 points)**

Solution:

$$\begin{bmatrix} 5 & 2 & 1 \\ 15/2 & 5/2 & 3/2 \\ 3 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 0 & 4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Subspaces (10 points)

Recall that a subset S of a vector space is called a subspace if two conditions hold:

(i) For every vector $\vec{v}$ in S, every $c\vec{v}$ is still in S.

(ii) For every two vectors $\vec{v}$ and $\vec{w}$ in S, the sum $\vec{v}+\vec{w}$ is still in S.

For each S defined below, state whether condition (i) holds, whether condition (ii) holds, and whether S is a subspace. (You do not need to show any other work.)

(a) The line S in $\mathbb{R}^2$ with equation $2x+4y=0$. **(2 points)**

Solution: (i) holds, (ii) holds, S is a subspace.

(b) The set S of all solutions $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ to the equation $\begin{bmatrix} 1 & -2 \\ -7 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \end{bmatrix}$. **(2 points)**

Solution: (i) does not hold, (ii) does not hold, S is not a subspace.

(c) The set S of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ such that x and y are integers. **(2 points)**

Solution: (i) does not hold, (ii) holds, S is not a subspace.

(d) The union S of the coordinate axes in $\mathbb{R}^2$ (the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x=0$ or $y=0$). **(2 points)**

Solution: (i) holds, (ii) does not hold, S is not a subspace.

(e) The set $S=\text{span}(\vec{v}_1,\ldots,\vec{v}_n)$, where $\vec{v}_1,\ldots,\vec{v}_n$ are vectors in a vector space V. **(2 points)**

Solution: (i) holds, (ii) holds, S is a subspace.

4 Inventions (10 points)

(a) Invent two vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^3$ so that $\text{span}(\vec{v}_1, \vec{v}_2)$ is a line. **(2 points)**

Possible solution: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$.

(b) Invent two vectors $\vec{v}$ and $\vec{w}$ so that $\vec{v}\vec{w}^T = \begin{bmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}$. **(2 points)**

Possible solution: $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(c) Invent a system of two linear equations in x and y that has no solution. **(2 points)**

Possible solution: $x + y = 0$, $x + y = 3$.

(d) Invent a 2×2 matrix A such that $N(A) = C(A)$. **(2 points)**

Possible solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(e) Invent a matrix A so that $\det(A) = -1$ and $\det(3A) = -27$. **(2 points)**

Possible solution: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5 Closest Line to Three Points (10 points)

Let $b = C + Dt$ be the equation for a line L in $\mathbb{R}^2$.

(a) Write down the three linear equations in C and D that would have to hold for L to pass through the points $\begin{bmatrix} t \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. **(2 points)**

Solution: $-2 = C$, $0 = C + D$, $0 = C + 2D$.

(b) Convert those linear equations into a matrix equation $A \begin{bmatrix} C \\ D \end{bmatrix} = \vec{b}$. **(1 point)**

Solution: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

(c) Write down the new matrix equation $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T \vec{b}$. **(2 points)**

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

(d) Solve the new matrix equation. **(3 points)**

Solution: $\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -10 \\ 6 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 1 \end{bmatrix}$

(e) Draw a graph containing the three points and the closest line L. **(2 points)**

Description of solution: Plot the three points and the line L with equation $b = -5/3 + t$.

6 Projection (10 points)

Let S be the plane in $\mathbb{R}^3$ spanned by vectors $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Compute an orthogonal basis $\vec{A}, \vec{B}$ for S. **(3 points)**

Possible solution:

$$\vec{A} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \qquad \vec{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(b) Normalize your vectors $\vec{A}$ and $\vec{B}$ to get an orthonormal basis $\vec{q}_1$, $\vec{q}_2$ for S. **(1 point)**

Solution:

$$\vec{q}_1 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \vec{q}_2 = \tfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) Compute the matrix $P = QQ^T$ that projects vectors orthogonally onto S . **(4 points)**

Solution:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}$$

(d) Show that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of P. What is the eigenvalue λ? **(2 points)**

Solution:

$$\begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/6 + 1/6 \\ 1/3 - 1/3 \\ -1/6 - 5/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \lambda = 1.$$

7 Differential Equations (10 points)

Consider the following population model. Let $b(t)$ denote the population of bananas and $g(t)$ the population of gorillas at time t. The growth rates of the two populations are

$$\frac{db}{dt} = 4b - 10g \qquad \text{and} \qquad \frac{dg}{dt} = \tfrac{1}{5}b + g.$$

(a) Write these growth rates as a differential equation of the form $\frac{d\vec{u}}{dt} = A\vec{u}$. **(2 points)**

(b) Compute the eigenvalues λ_1, λ_2 and corresponding eigenvectors $\vec{x}_1$, $\vec{x}_2$ of A. **(5 points)**

(c) Initially, there are $b(0) = 60$ bananas and $g(0) = 10$ gorillas. Compute the scalars C_1, C_2 that give the unique solution $\vec{u}(t) = C_1 e^{\lambda_1 t}\vec{x}_1 + C_2 e^{\lambda_2 t}\vec{x}_2$. **(3 points)**

8 Diagonalization (10 points)

Let $A = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

(a) Compute the eigenvalues λ_1, λ_2 of A. **(2 points)**

Solution:

$$\begin{vmatrix} -1-\lambda & 4 \\ -2 & 5-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 + 8 = (\lambda - 3)(\lambda - 1), \qquad \lambda_1 = 3, \lambda_2 = 1.$$

(b) Compute independent eigenvectors $\vec{x}_1$, $\vec{x}_2$ of A. **(3 points)**

Solution:

$$A - 3I = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \qquad \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(c) Write down the factorization $A = S\Lambda S^{-1}$, where S is a matrix of eigenvectors and Λ is the eigenvalue matrix. **(2 points)**

Solution:

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

(d) Use your answer in (c) to compute A^4. Simplify completely. **(3 points)**

Solution:

$$A^4 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 81 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 81 & 2 \\ 81 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -79 & 160 \\ -80 & 161 \end{bmatrix}$$

9 Linear Transformations in $\mathbb{R}^2$ (10 points)

Suppose you know $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ is linear and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}7\\-3\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}6\\1\end{bmatrix}$.

(a) Compute each of the following if you can, or state that not enough information is given: **(3 points)**

Solution:

(i) $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = T\left(0\begin{bmatrix}2\\1\end{bmatrix}\right) = 0\,T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = 0\begin{bmatrix}7\\-3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

(ii) $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\1\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}7\\-3\end{bmatrix} - \begin{bmatrix}6\\1\end{bmatrix} = \begin{bmatrix}1\\-4\end{bmatrix}$

(iii) $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(-\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}\right) = -\begin{bmatrix}7\\-3\end{bmatrix} + 2\begin{bmatrix}6\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$

(b) T acts as multiplication by a matrix A. Use your answer in (a) to find A. **(2 points)**

Solution: $A = \begin{bmatrix}1 & 5\\-4 & 5\end{bmatrix}$

(c) Draw the square with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ and draw the parallelogram that you get when T transforms that square. **(4 points)**

Description of solution: The parallelogram should have vertices at $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\-4\end{bmatrix}$, $\begin{bmatrix}5\\5\end{bmatrix}$, and $\begin{bmatrix}6\\1\end{bmatrix}$.

(d) Compute $\det A$. This is the area of the parallelogram you just drew! **(1 point)**

Solution: $\det A = 5 + 20 = 25$

10 More Inventions (10 points)

(a) Invent a 2×2 matrix A that has an eigenvalue of multiplicity 2 but only one independent eigenvector. **(2 points)**

Possible solution: $A = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$.

(b) Invent a matrix A such that no other matrix is similar to A. **(2 points)**

Possible solution: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) Invent a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ such that the $\mathcal{B}$-coordinates of the vector $\begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. **(2 points)**

Possible solution: $\vec{v}_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

(d) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$. Invent a matrix M that changes the coordinates of vectors from standard coordinates to $\mathcal{B}$-coordinates. **(2 points)**

Possible solution: $M = \frac{1}{7} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

(e) Let V be the vector space of all polynomials in x of degree ≤ 2. Invent a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of V such that the $\mathcal{B}$-coordinates of $x + x^2$ are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. **(2 points)**

Possible solution: $\vec{v}_1 = 1$, $\vec{v}_2 = -1 + x$, $\vec{v}_3 = x^2$.