Final Exam

1 The Four Subspaces (10 points)

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$.

(a) Compute the rank r of A. **(1 point)**

(b) Use r to compute the dimensions of the four fundamental subspaces $N(A)$, $C(A)$, $C(A^T)$, $N(A^T)$. **(2 points)**

(c) Which pairs of subspaces are orthogonal? **(1 point)**

(d) Compute bases for the four fundamental subspaces of A. **(6 points)**

2 Inverse (10 points)

Let $A = \begin{bmatrix} -1 & 0 & 2 \\ 3 & -2 & 0 \\ 0 & 4 & -10 \end{bmatrix}$.

(a) Compute the inverse A^{-1} of A using Gauss-Jordan elimination on $\begin{bmatrix} A & I \end{bmatrix}$ or the cofactor formula $\frac{1}{\det A} C^T$. **(8 points)**

(b) Check your inverse is correct by showing $A^{-1}A = I$. **(2 points)**

3 Subspaces (10 points)

Recall that a subset S of a vector space is called a subspace if two conditions hold:

(i) For every vector $\vec{v}$ in S, every $c\vec{v}$ is still in S.

(ii) For every two vectors $\vec{v}$ and $\vec{w}$ in S, the sum $\vec{v} + \vec{w}$ is still in S.

For each S defined below, state whether condition (i) holds, whether condition (ii) holds, and whether S is a subspace. (You do not need to show any other work.)

(a) The line S in $\mathbb{R}^2$ with equation $2x + 4y = 0$. **(2 points)**

(b) The set S of all solutions $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ to the equation $\begin{bmatrix} 1 & -2 \\ -7 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \end{bmatrix}$. **(2 points)**

(c) The set S of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ such that x and y are integers. **(2 points)**

(d) The union S of the coordinate axes in $\mathbb{R}^2$ (the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x = 0$ or $y = 0$). **(2 points)**

(e) The set $S = \text{span}(\vec{v}_1, \ldots, \vec{v}_n)$, where $\vec{v}_1, \ldots, \vec{v}_n$ are vectors in a vector space V. **(2 points)**

4 Inventions (10 points)

(a) Invent two vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^3$ so that $\text{span}(\vec{v}_1, \vec{v}_2)$ is a line. **(2 points)**

(b) Invent two vectors $\vec{v}$ and $\vec{w}$ so that $\vec{v}\vec{w}^T = \begin{bmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}$. **(2 points)**

(c) Invent a system of two linear equations in x and y that has no solution. **(2 points)**

(d) Invent a 2×2 matrix A such that $N(A) = C(A)$. **(2 points)**

(e) Invent a matrix A so that $\det(A) = -1$ and $\det(3A) = -27$. **(2 points)**

5 Closest Line to Three Points (10 points)

Let $b = C + Dt$ be the equation for a line L in $\mathbb{R}^2$.

(a) Write down the three linear equations in C and D that would have to hold for L to pass through the points $\begin{bmatrix} t \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. **(2 points)**

(b) Convert those linear equations into a matrix equation $A \begin{bmatrix} C \\ D \end{bmatrix} = \vec{b}$. **(1 point)**

(c) Write down the new matrix equation $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T \vec{b}$. **(2 points)**

(d) Solve the new matrix equation. **(3 points)**

(e) Draw a graph containing the three points and the closest line L. **(2 points)**

6 Projection (10 points)

Let S be the plane in $\mathbb{R}^3$ spanned by vectors $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Compute an orthogonal basis $\vec{A}, \vec{B}$ for S. **(3 points)**

(b) Normalize your vectors $\vec{A}$ and $\vec{B}$ to get an orthonormal basis $\vec{q}_1$, $\vec{q}_2$ for S. **(1 point)**

(c) Compute the matrix $P = QQ^T$ that projects vectors orthogonally onto S . **(4 points)**

(d) Show that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of P. What is the eigenvalue λ? **(2 points)**

7 Differential Equations (10 points)

Consider the following population model. Let $b(t)$ denote the population of bananas and $g(t)$ the population of gorillas at time t. The growth rates of the two populations are

$$\frac{db}{dt} = 4b - 10g \qquad \text{and} \qquad \frac{dg}{dt} = \tfrac{1}{5}b + g.$$

(a) Write these growth rates as a differential equation of the form $\frac{d\vec{u}}{dt} = A\vec{u}$. **(2 points)**

(b) Compute the eigenvalues λ_1, λ_2 and corresponding eigenvectors $\vec{x}_1$, $\vec{x}_2$ of A. **(5 points)**

(c) Initially, there are $b(0) = 60$ bananas and $g(0) = 10$ gorillas. Compute the scalars C_1, C_2 that give the unique solution $\vec{u}(t) = C_1 e^{\lambda_1 t}\vec{x}_1 + C_2 e^{\lambda_2 t}\vec{x}_2$. **(3 points)**

8 Diagonalization (10 points)

Let $A = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

(a) Compute the eigenvalues λ_1, λ_2 of A. **(2 points)**

(b) Compute independent eigenvectors $\vec{x}_1$, $\vec{x}_2$ of A. **(3 points)**

(c) Write down the factorization $A = S\Lambda S^{-1}$, where S is a matrix of eigenvectors and Λ is the eigenvalue matrix. **(2 points)**

(d) Use your answer in (c) to compute A^4. Simplify completely. **(3 points)**

9 Linear Transformations in $\mathbb{R}^2$ (10 points)

Suppose you know $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ is linear and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}7\\-3\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}6\\1\end{bmatrix}$.

(a) Compute each of the following if you can, or state that not enough information is given: **(3 points)**

(i) $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right)$

(ii) $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

(iii) $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

(b) T acts as multiplication by a matrix A. Use your answer in (a) to find A. **(2 points)**

(c) Draw the square with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ and draw the parallelogram that you get when T transforms that square. **(4 points)**

(d) Compute $\det A$. This is the area of the parallelogram you just drew! **(1 point)**

10 More Inventions (10 points)

(a) Invent a 2×2 matrix A that has an eigenvalue of multiplicity 2 but only one independent eigenvector. **(2 points)**

(b) Invent a matrix A such that no other matrix is similar to A. **(2 points)**

(c) Invent a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ such that the $\mathcal{B}$-coordinates of the vector $\begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. **(2 points)**

(d) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$. Invent a matrix M that changes the coordinates of vectors from standard coordinates to $\mathcal{B}$-coordinates. **(2 points)**

(e) Let V be the vector space of all polynomials in x of degree ≤ 2. Invent a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of V such that the $\mathcal{B}$-coordinates of $x + x^2$ are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. **(2 points)**