seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.

- **Linear programming.** Many important management decisions today are made on the basis of linear programming models that utilize hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.

- **Electrical networks.** Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

---

1.1 **SYSTEMS OF LINEAR EQUATIONS**

A linear equation in the variables \( x_1, \ldots, x_n \) is an equation that can be written in the form

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b
\]

where \( b \) and the coefficients \( a_1, \ldots, a_n \) are real or complex numbers, usually known in advance. The subscript \( n \) may be any positive integer. In textbook examples and exercises, \( n \) is normally between 2 and 5. In real-life problems, \( n \) might be 50 or 5000, or even larger.

The equations

\[
4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3
\]

are both linear because they can be rearranged algebraically as in equation (1):

\[
3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}
\]

The equations

\[
4x_1 - 5x_2 = x_1 x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6
\]

are not linear because of the presence of \( x_1 x_2 \) in the first equation and \( \sqrt{x_1} \) in the second.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables—say, \( x_1, \ldots, x_n \). An example is

\[
\begin{align*}
2x_1 - x_2 + 1.5x_3 &= 8 \\
x_1 - 4x_3 &= -7
\end{align*}
\]
A solution of the system is a list \((s_1, s_2, \ldots, s_n)\) of numbers that makes each equation a true statement when the values \(s_1, \ldots, s_n\) are substituted for \(x_1, \ldots, x_n\), respectively. For instance, \((5, 6.5, 3)\) is a solution of system (2) because, when these values are substituted in (2) for \(x_1, x_2, x_3\), respectively, the equations simplify to \(8 = 8\) and \(-7 = -7\).

The set of all possible solutions is called the solution set of the linear system. Two linear systems are called equivalent if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

\[
\begin{align*}
    x_1 - 2x_2 &= -1 \\
    -x_1 + 3x_2 &= 3
\end{align*}
\]

The graphs of these equations are lines, which we denote by \(\ell_1\) and \(\ell_2\). A pair of numbers \((x_1, x_2)\) satisfies both equations in the system if and only if the point \((x_1, x_2)\) lies on both \(\ell_1\) and \(\ell_2\). In the system above, the solution is the single point \((3, 2)\), as you can easily verify. See Fig. 1.

![Figure 1](image1.png)

**FIGURE 1** Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence “intersect” at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

\[
\begin{align*}
    \text{(a)} \quad x_1 - 2x_2 &= -1 \\
    \quad -x_1 + 3x_2 &= 3 \\
    \text{(b)} \quad x_1 - 2x_2 &= -1 \\
    \quad -x_1 + 2x_2 &= 1
\end{align*}
\]

![Figure 2](image2.png)

**FIGURE 2** (a) No solution. (b) Infinitely many solutions.

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.
A system of linear equations has
1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

**Matrix Notation**

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

with the coefficients of each variable aligned in columns, the matrix

\[
\begin{bmatrix}
1 & -2 & 1 \\
0 & 2 & -8 \\
-4 & 5 & 9
\end{bmatrix}
\]

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{bmatrix}
\]

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as \(0 \cdot x_1 + 2x_2 - 8x_3 = 8\).) An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

The **size** of a matrix tells how many rows and columns it has. The augmented matrix (4) above has 3 rows and 4 columns and is called a \(3 \times 4\) (read “3 by 4”) matrix. If \(m\) and \(n\) are positive integers, an \(m \times n\) **matrix** is a rectangular array of numbers with \(m\) rows and \(n\) columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

**Solving a Linear System**

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is *to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.*

Roughly speaking, use the \(x_1\) term in the first equation of a system to eliminate the \(x_1\) terms in the other equations. Then use the \(x_2\) term in the second equation to eliminate the \(x_2\) terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system: Replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. After the first example, you will see why these three operations do not change the solution set of the system.
**EXAMPLE 1** Solve system (3).

**SOLUTION** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison:

\[
\begin{align*}
&x_1 - 2x_2 + x_3 = 0 \\
&2x_2 - 8x_3 = 8 \\
&-4x_1 + 5x_2 + 9x_3 = -9
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{bmatrix}
\]

*Keep \( x_1 \) in the first equation and eliminate it from the other equations.* To do so, add 4 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

\[
\frac{4 \cdot [\text{equation 1}]}{+ \ [\text{equation 3}]} \quad \frac{4x_1 - 8x_2 + 4x_3 = 0}{-4x_1 + 5x_2 + 9x_3 = -9} \quad [\text{new equation 3}]: \quad -3x_2 + 13x_3 = -9
\]

The result of this calculation is written in place of the original third equation:

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-3x_2 + 13x_3 &= -9
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{bmatrix}
\]

Now, multiply equation 2 by 1/2 in order to obtain 1 as the coefficient for \( x_2 \). (This calculation will simplify the arithmetic in the next step.)

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
-3x_2 + 13x_3 &= -9
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & -3 & 13 & -9
\end{bmatrix}
\]

Use the \( x_2 \) in equation 2 to eliminate the \(-3x_2\) in equation 3. The “mental” computation is

\[
\frac{3 \cdot [\text{equation 2}]}{+ \ [\text{equation 3}]} \quad \frac{3x_2 - 12x_3 = 12}{-3x_2 + 13x_3 = -9} \quad [\text{new equation 3}]: \quad x_3 = 3
\]

The new system has a triangular form:

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Eventually, you want to eliminate the \(-2x_2\) term from equation 1, but it is more efficient to use the \( x_3 \) in equation 3 first, to eliminate the \(-4x_3\) and \(+x_3\) terms in equations 2 and 1. The two “mental” calculations are

\[
\begin{align*}
4 \cdot [\text{eq. 3}]: \quad 4x_3 &= 12 \\
+ \ [\text{eq. 2}]: \quad x_2 - 4x_3 &= 4 \\
[\text{new eq. 2}]: \quad x_2 &= 16
\end{align*}
\]

\[
\begin{align*}
-1 \cdot [\text{eq. 3}]: \quad -x_3 &= -3 \\
+ \ [\text{eq. 1}]: \quad x_1 - 2x_2 + x_3 &= 0 \\
[\text{new eq. 1}]: \quad x_1 - 2x_2 &= -3
\end{align*}
\]

\[\text{The intuitive term } \text{triangular} \text{ will be replaced by a precise term in the next section.}\]
It is convenient to combine the results of these two operations:

\[
\begin{align*}
x_1 - 2x_2 &= -3 \\
x_2 &= 16 \\
x_3 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 0 & -3 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Now, having cleaned out the column above the \(x_3\) in equation 3, move back to the \(x_2\) in equation 2 and use it to eliminate the \(-2x_2\) above it. Because of the previous work with \(x_3\), there is now no arithmetic involving \(x_3\) terms. Add 2 times equation 2 to equation 1 and obtain the system:

\[
\begin{align*}
x_1 &= 29 \\
x_2 &= 16 \\
x_3 &= 3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

The work is essentially done. It shows that the only solution of the original system is \((29, 16, 3)\). However, since there are so many calculations involved, it is a good practice to check the work. To verify that \((29, 16, 3)\) is a solution, substitute these values into the left side of the original system, and compute:

\[
\begin{align*}
(29) - 2(16) + (3) &= 29 - 32 + 3 = 0 \\
2(16) - 8(3) &= 32 - 24 = 8 \\
-4(29) + 5(16) + 9(3) &= -116 + 80 + 27 = -9
\end{align*}
\]

The results agree with the right side of the original system, so \((29, 16, 3)\) is a solution of the system. \(\blacksquare\)

Example 1 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

**ELEMENTARY ROW OPERATIONS**

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.\(^2\)

2. (Interchange) Interchange two rows.

3. (Scaling) Multiply all entries in a row by a nonzero constant.

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange. If a row is scaled by a nonzero constant \(c\), then multiplying the new row by \(1/c\) produces the original row. Finally, consider a replacement operation involving two rows—say, rows 1 and 2—and suppose that \(c\) times row 1 is added to row 2 to produce a new row. To “reverse” this operation, add \(-c\) times row 1 to (new) row 2 and obtain the original row 2. See Exercises 29–32 at the end of this section.

\(^2\)A common paraphrase of row replacement is “Add to one row a multiple of another row.”
At the moment, we are interested in row operations on the augmented matrix of a system of linear equations. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Though Example 1 is lengthy, you will find that after some practice, the calculations go quickly. Row operations in the text and exercises will usually be extremely easy to perform, allowing you to focus on the underlying concepts. Still, you must learn to perform row operations accurately because they will be used throughout the text.

The rest of this section shows how to use row operations to determine the size of a solution set, without completely solving the linear system.

**Existence and Uniqueness Questions**

Section 1.2 will show why a solution set for a linear system contains either no solutions, one solution, or infinitely many solutions. Answers to the following two questions will determine the nature of the solution set for a linear system.

To determine which possibility is true for a particular system, we ask two questions.

**TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM**

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

These two questions will appear throughout the text, in many different guises. This section and the next will show how to answer these questions via row operations on the augmented matrix.

**EXAMPLE 2** Determine if the following system is consistent:

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

**SOLUTION** This is the system from Example 1. Suppose that we have performed the row operations necessary to obtain the triangular form

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{align*}
\]

At this point, we know \(x_3\). Were we to substitute the value of \(x_3\) into equation 2, we could compute \(x_2\) and hence could determine \(x_1\) from equation 1. So a solution exists; the system is consistent. (In fact, \(x_2\) is uniquely determined by equation 2 since \(x_3\) has
only one possible value, and \( x_1 \) is therefore uniquely determined by equation 1. So the solution is unique.)

**EXAMPLE 3** Determine if the following system is consistent:

\[
\begin{align*}
x_2 - 4x_3 &= 8 \\
2x_1 - 3x_2 + 2x_3 &= 1 \\
5x_1 - 8x_2 + 7x_3 &= 1
\end{align*}
\]

**SOLUTION** The augmented matrix is

\[
\begin{bmatrix}
0 & 1 & -4 & 8 \\
2 & -3 & 2 & 1 \\
5 & -8 & 7 & 1
\end{bmatrix}
\]

To obtain an \( x_1 \) in the first equation, interchange rows 1 and 2:

\[
\begin{bmatrix}
2 & -3 & 2 & 1 \\
0 & 1 & -4 & 8 \\
5 & -8 & 7 & 1
\end{bmatrix}
\]

To eliminate the \( 5x_1 \) term in the third equation, add \(-5/2\) times row 1 to row 3:

\[
\begin{bmatrix}
2 & -3 & 2 & 1 \\
0 & 1 & -4 & 8 \\
0 & -1/2 & 2 & -3/2
\end{bmatrix}
\]

Next, use the \( x_2 \) term in the second equation to eliminate the \(-1/2x_2\) term from the third equation. Add \(1/2\) times row 2 to row 3:

\[
\begin{bmatrix}
2 & -3 & 2 & 1 \\
0 & 1 & -4 & 8 \\
0 & 0 & 0 & 5/2
\end{bmatrix}
\]

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

\[
\begin{align*}
2x_1 - 3x_2 + 2x_3 &= 1 \\
x_2 - 4x_3 &= 8 \\
0 &= 5/2
\end{align*}
\]

The equation \(0 = 5/2\) is a short form of \(0x_1 + 0x_2 + 0x_3 = 5/2\). This system in triangular form obviously has a built-in contradiction. There are no values of \( x_1, x_2, x_3 \) that satisfy (8) because the equation \(0 = 5/2\) is never true. Since (8) and (5) have the same solution set, the original system is inconsistent (i.e., has no solution).

Pay close attention to the augmented matrix in (7). Its last row is typical of an inconsistent system in triangular form.
NUMERICAL NOTE

In real-world problems, systems of linear equations are solved by a computer. For a square coefficient matrix, computer programs nearly always use the elimination algorithm given here and in Section 1.2, modified slightly for improved accuracy. The vast majority of linear algebra problems in business and industry are solved with programs that use floating point arithmetic. Numbers are represented as decimals $d_1 \cdots d_p \times 10^r$, where $r$ is an integer and the number $p$ of digits to the right of the decimal point is usually between 8 and 16. Arithmetic with such numbers typically is inexact, because the result must be rounded (or truncated) to the number of digits stored. “Roundoff error” is also introduced when a number such as 1/3 is entered into the computer, since its decimal representation must be approximated by a finite number of digits. Fortunately, inaccuracies in floating point arithmetic seldom cause problems. The numerical notes in this book will occasionally warn of issues that you may need to consider later in your career.

PRACTICE PROBLEMS

Throughout the text, practice problems should be attempted before working the exercises. Solutions appear after each exercise set.

1. State in words the next elementary row operation that should be performed on the system in order to solve it. [More than one answer is possible in (a).]
   a. $x_1 + 4x_2 - 2x_3 + 8x_4 = 12$
   b. $x_1 - 3x_2 + 5x_3 - 2x_4 = 0$
   $x_2 - 7x_3 + 2x_4 = -4$
   $5x_3 - x_4 = 7$
   $x_3 + 3x_4 = -5$

2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

\[
\begin{bmatrix}
1 & 5 & 2 & -6 \\
0 & 4 & -7 & 2 \\
0 & 0 & 5 & 0
\end{bmatrix}
\]

3. Is $(3, 4, -2)$ a solution of the following system?

\[
\begin{align*}
5x_1 - x_2 + 2x_3 &= 7 \\
-2x_1 + 6x_2 + 9x_3 &= 0 \\
-7x_1 + 5x_2 - 3x_3 &= -7
\end{align*}
\]

4. For what values of $h$ and $k$ is the following system consistent?

\[
\begin{align*}
2x_1 - x_2 &= h \\
-6x_1 + 3x_2 &= k
\end{align*}
\]
1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. \[x_1 + 5x_2 = 7\]
2. \[3x_1 + 6x_2 = -3\]
   \[-2x_1 - 7x_2 = -5\]
   \[5x_1 + 7x_2 = 10\]

3. Find the point \((x_1, x_2)\) that lies on the line \(x_1 + 2x_2 = 4\) and on the line \(x_1 - x_2 = 1\). See the figure.

![Diagram](image)

4. Find the point of intersection of the lines \(x_1 + 2x_2 = -13\) and \(3x_1 - 2x_2 = 1\)

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. \[
\begin{bmatrix}
1 & -4 & -3 & 0 & 7 \\
0 & 1 & 4 & 0 & 6 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & -5
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & -6 & 4 & 0 & -1 \\
0 & 2 & -7 & 0 & 4 \\
0 & 0 & 1 & 2 & -3 \\
0 & 0 & 4 & 1 & 2
\end{bmatrix}
\]

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. \[
\begin{bmatrix}
1 & 7 & 3 & -4 \\
0 & 1 & -1 & 3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & -5 & 4 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 2 & 0
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & -1 & 0 & 0 & -5 \\
0 & 1 & -2 & 0 & -7 \\
0 & 0 & 1 & -3 & 2 \\
0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
1 & 3 & 0 & -2 & -7 \\
0 & 1 & 0 & 3 & 6 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & -2
\end{bmatrix}
\]

Solve the systems in Exercises 11–14.

11. \[x_2 + 5x_3 = -4\]
    \[x_1 + 4x_2 + 3x_3 = -2\]
    \[2x_1 + 7x_2 + x_3 = -2\]

12. \[x_1 - 5x_2 + 4x_3 = -3\]
    \[2x_1 - 7x_2 + 3x_3 = -2\]
    \[-2x_1 + x_2 + 7x_3 = -1\]

13. \[x_1 - 3x_3 = 8\]
    \[2x_1 + 2x_2 + 9x_3 = 7\]
    \[x_2 + 5x_3 = -2\]

14. \[2x_1 - 6x_3 = -8\]
    \[x_2 + 2x_3 = 3\]
    \[3x_1 + 6x_2 - 2x_3 = -4\]

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

15. \[x_1 - 6x_2 = 5\]
    \[x_2 - 4x_3 + x_4 = 0\]
    \[-x_1 + 6x_2 + x_3 + 5x_4 = 3\]
    \[-x_2 + 5x_3 + 4x_4 = 0\]

16. \[2x_1 - 4x_2 = -10\]
    \[3x_2 + 3x_3 = 0\]
    \[x_3 + 4x_4 = -1\]
    \[-3x_1 + 2x_2 + 3x_3 + x_4 = 5\]

17. Do the three lines \(2x_1 + 3x_2 = -1,\ \ 6x_1 + 5x_2 = 0,\ \text{and}\ \ 2x_1 - 5x_2 = 7\) have a common point of intersection? Explain.

18. Do the three planes \(2x_1 + 4x_2 + 4x_3 = 4,\ \ x_2 - 2x_3 = -2,\ \text{and}\ \ x_1 + 3x_3 = 0\) have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of \(h\) such that the matrix is the augmented matrix of a consistent linear system.

19. \[
\begin{bmatrix}
1 & h & 4 \\
3 & 6 & 8
\end{bmatrix}
\]

20. \[
\begin{bmatrix}
1 & h & -5 \\
2 & -8 & 6
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
1 & 4 & -2 \\
3 & h & -6
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
-4 & 12 & h \\
2 & -6 & -3
\end{bmatrix}
\]

In Exercises 23 and 24, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the
approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

23. a. Every elementary row operation is reversible.
   b. A $5 \times 6$ matrix has six rows.
   c. The solution set of a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.
   d. Two fundamental questions about a linear system involve existence and uniqueness.

24. a. Two matrices are row equivalent if they have the same number of rows.
   b. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.
   c. Two equivalent linear systems can have different solution sets.
   d. A consistent system of linear equations has one or more solutions.

25. Find an equation involving $g$, $h$, and $k$ that makes this augmented matrix correspond to a consistent system:

   $\begin{bmatrix}
   1 & -4 & 7 & g \\
   0 & 3 & -5 & h \\
   -2 & 5 & -9 & k
   \end{bmatrix}$

26. Suppose the system below is consistent for all possible values of $f$ and $g$. What can you say about the coefficients $c$ and $d$? Justify your answer.

   $2x_1 + 4x_2 = f$
   $cx_1 + dx_2 = g$

27. Suppose $a$, $b$, $c$, and $d$ are constants such that $a$ is not zero and the system below is consistent for all possible values of $f$ and $g$. What can you say about the numbers $a$, $b$, $c$, and $d$? Justify your answer.

   $ax_1 + bx_2 = f$
   $cx_1 + dx_2 = g$

28. Construct three different augmented matrices for linear systems whose solution set is $x_1 = 3$, $x_2 = -2$, $x_3 = -1$.

In Exercises 29–32, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

29. $\begin{bmatrix}
   0 & -2 & 5 \\
   1 & 3 & -5 \\
   3 & -1 & 6 \\
\end{bmatrix}$, $\begin{bmatrix}
   3 & -1 & 6 \\
   1 & 3 & -5 \\
   0 & -2 & 5 \\
\end{bmatrix}$

30. $\begin{bmatrix}
   0 & -2 & 6 \\
   1 & 3 & -4 \\
   0 & -5 & 10 \\
\end{bmatrix}$, $\begin{bmatrix}
   1 & 3 & -4 \\
   0 & -2 & 6 \\
   0 & 1 & -2 \\
\end{bmatrix}$

31. $\begin{bmatrix}
   1 & -2 & 1 \\
   0 & 5 & -2 \\
   4 & -1 & 3 \\
\end{bmatrix}$, $\begin{bmatrix}
   1 & -2 & 1 \\
   0 & 5 & -2 \\
   7 & 1 & -1 \\
\end{bmatrix}$

32. $\begin{bmatrix}
   1 & 2 & -5 \\
   0 & 1 & -3 \\
   0 & 4 & -12 \\
\end{bmatrix}$, $\begin{bmatrix}
   1 & 2 & -5 \\
   0 & 1 & -3 \\
   0 & 0 & 15 \\
\end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.\footnote{See Frank M. White, Heat and Mass Transfer (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.}

33. Write a system of four equations whose solution gives estimates for the temperatures $T_1, \ldots, T_4$.

34. Solve the system of equations from Exercise 33. \textit{[Hint: To speed up the calculations, interchange rows 1 and 4 before starting “replace” operations.]}

---

**Solutions to Practice Problems**

1. a. For “hand computation,” the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by 1/5. Or, replace equation 4 by its sum with $-1/5$ times row 3. (In any case, do not use the $x_2$ in equation 2 to eliminate the $4x_2$ in equation 1. Wait until a triangular form has been reached and the $x_3$ terms and $x_4$ terms have been eliminated from the first two equations.)
b. The system is in triangular form. Further simplification begins with the \( x_4 \) in the fourth equation. Use the \( x_4 \) to eliminate all \( x_4 \) terms above it. The appropriate step now is to add 2 times equation 4 to equation 1. (After that, move to equation 3, multiply it by 1/2, and then use the equation to eliminate the \( x_3 \) terms above it.\)

2. The system corresponding to the augmented matrix is

\[
\begin{align*}
  x_1 + 5x_2 + 2x_3 &= -6 \\
  4x_2 - 7x_3 &= 2 \\
  5x_3 &= 0
\end{align*}
\]

The third equation makes \( x_3 = 0 \), which is certainly an allowable value for \( x_3 \). After eliminating the \( x_3 \) terms in equations 1 and 2, you could go on to solve for unique values for \( x_2 \) and \( x_1 \). Hence a solution exists, and it is unique. Contrast this situation with that in Example 3.

3. It is easy to check if a specific list of numbers is a solution. Set \( x_1 = 3 \), \( x_2 = 4 \), and \( x_3 = -2 \), and find that

\[
\begin{align*}
  5(3) - (4) + 2(-2) &= 15 - 4 - 4 = 7 \\
  -2(3) + 6(4) + 9(-2) &= -6 + 24 - 18 = 0 \\
  -7(3) + 5(4) - 3(-2) &= -21 + 20 + 6 = 5
\end{align*}
\]

Although the first two equations are satisfied, the third is not, so \( (3, 4, -2) \) is not a solution of the system. Notice the use of parentheses when making the substitutions. They are strongly recommended as a guard against arithmetic errors.

4. When the second equation is replaced by its sum with 3 times the first equation, the system becomes

\[
\begin{align*}
  2x_1 - x_2 &= h \\
  0 &= k + 3h
\end{align*}
\]

If \( k + 3h \) is nonzero, the system has no solution. The system is consistent for any values of \( h \) and \( k \) that make \( k + 3h = 0 \).

### 1.2 Row Reduction and Echelon Forms

This section refines the method of Section 1.1 into a row reduction algorithm that will enable us to analyze any system of linear equations.\(^1\) By using only the first part of the algorithm, we will be able to answer the fundamental existence and uniqueness questions posed in Section 1.1.

The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system. So the first part of this section concerns an arbitrary rectangular matrix and begins by introducing two important classes of matrices that include the “triangular” matrices of Section 1.1. In the definitions that follow, a nonzero row or column in a matrix means a row or column that contains at least one nonzero entry; a leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

\(^1\)The algorithm here is a variant of what is commonly called \textit{Gaussian elimination}. A similar elimination method for linear systems was used by Chinese mathematicians in about 250 B.C. The process was unknown in Western culture until the nineteenth century, when a famous German mathematician, Carl Friedrich Gauss, discovered it. A German engineer, Wilhelm Jordan, popularized the algorithm in an 1888 text on geodesy.
DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an echelon (“steplike”) pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The “triangular” matrices of Section 1.1, such as

\[
\begin{bmatrix}
2 & -3 & 2 & 1 \\
0 & 1 & -4 & 8 \\
0 & 0 & 0 & 5/2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

**EXAMPLE 1** The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries (*) may have any value (including zero).

\[
\begin{bmatrix}
\text{■} & \ast & \ast & \ast \\
0 & \text{■} & \ast & \ast \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & \text{■} & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & \text{■} & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \text{■} & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & \text{■} & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & \text{■} & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & \ast
\end{bmatrix}
\]

The following matrices are in reduced echelon form because the leading entries are 1’s, and there are 0’s below and above each leading 1.

\[
\begin{bmatrix}
1 & 0 & \ast & \ast \\
0 & 1 & \ast & \ast \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 1 & \ast & 0 & 0 & \ast & \ast & 0 & \ast \\
0 & 0 & 1 & 0 & 0 & \ast & \ast & 0 & \ast \\
0 & 0 & 0 & 1 & 0 & \ast & \ast & 0 & \ast \\
0 & 0 & 0 & 0 & 1 & \ast & \ast & 0 & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ast
\end{bmatrix}
\]

Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. The following theorem is proved in Appendix A at the end of the text.

**THEOREM 1**

**Uniqueness of the Reduced Echelon Form**

Each matrix is row equivalent to one and only one reduced echelon matrix.
If a matrix $A$ is row equivalent to an echelon matrix $U$, we call $U$ an echelon form (or row echelon form) of $A$; if $U$ is in reduced echelon form, we call $U$ the reduced echelon form of $A$. [Most matrix programs and calculators with matrix capabilities use the abbreviation RREF for reduced (row) echelon form. Some use REF for (row) echelon form.]

**Pivot Positions**

When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, the leading entries are always in the same positions in any echelon form obtained from a given matrix. These leading entries correspond to leading 1’s in the reduced echelon form.

**Definition**

A **pivot position** in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$. A **pivot column** is a column of $A$ that contains a pivot position.

In Example 1, the squares (●) identify the pivot positions. Many fundamental concepts in the first four chapters will be connected in one way or another with pivot positions in a matrix.

**Example 2** Row reduce the matrix $A$ below to echelon form, and locate the pivot columns of $A$.

$$A = \begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}$$

**Solution** Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely, in the second column. Choose the 2 in this position as the next pivot.

$$\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
0 & -3 & -6 & 4 & 9
\end{bmatrix}$$

(1)
Add \(-5/2\) times row 2 to row 3, and add \(3/2\) times row 2 to row 4.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 0 \\
\end{bmatrix}
\]

The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can’t use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of \(A\) are pivot columns.

A **pivot**, as illustrated in Example 2, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots in Example 2 were 1, 2, and \(-5\). Notice that these numbers are not the same as the actual elements of \(A\) in the highlighted pivot positions shown in (3).

With Example 2 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix. Careful study and mastery of this procedure now will pay rich dividends later in the course.

### The Row Reduction Algorithm

The algorithm that follows consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form. We illustrate the algorithm by an example.

**EXAMPLE 3** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
3 & -7 & 8 & -5 & 8 & 9 \\
3 & -9 & 12 & -9 & 6 & 15 \\
\end{bmatrix}
\]

**SOLUTION**

**STEP 1**

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
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\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
3 & -7 & 8 & -5 & 8 & 9 \\
3 & -9 & 12 & -9 & 6 & 15
\end{bmatrix}
\]

Pivot column

**STEP 2**
Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
3 & -7 & 8 & -5 & 8 & 9 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}
\]

**STEP 3**
Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3’s in column 1, it is just as easy to add $-1$ times row 1 to row 2.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}
\]

**STEP 4**
Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}
\]

New pivot column

For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the “top” row to the row below. This produces

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]
When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & \text{Pivot}
\end{bmatrix}
\]

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

**STEP 5**

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 2 & -4 & 4 & 0 & -14 \\
0 & 0 & 0 & 0 & 1 & \text{Pivot}
\end{bmatrix} \xrightarrow{\text{Row 1 + (-6) \cdot row 3}} \begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 2 & -4 & 4 & -6 & -6 \\
0 & 0 & 0 & 0 & 1 & \text{Pivot}
\end{bmatrix} \xrightarrow{\text{Row 2 + (-2) \cdot row 3}} \begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 2 & -4 & 4 & 0 & -14 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

The next pivot is in row 2. Scale this row, dividing by the pivot.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \xrightarrow{\text{Row scaled by } \frac{1}{4}} \begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 1 & -2 & 2 & 0 & -1 \frac{7}{4} \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

Create a zero in column 2 by adding 9 times row 2 to row 1.

\[
\begin{bmatrix}
3 & 0 & -6 & 9 & 0 & -72 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \xrightarrow{\text{Row 1 + (9) \cdot row 2}} \begin{bmatrix}
3 & 0 & -6 & 9 & 0 & -72 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

Finally, scale row 1, dividing by the pivot, 3.

\[
\begin{bmatrix}
1 & 0 & -2 & 3 & 0 & -24 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \xrightarrow{\text{Row scaled by } \frac{1}{3}} \begin{bmatrix}
1 & 0 & -2 & 3 & 0 & -24 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

This is the reduced echelon form of the original matrix.

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

**Numerical Note**

In step 2 above, a computer program usually selects as a pivot the entry in a column having the largest absolute value. This strategy, called **partial pivoting**, is used because it reduces roundoff errors in the calculations.
Solutions of Linear Systems

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system. Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

$$
\begin{bmatrix}
1 & 0 & -5 & 1 \\
0 & 1 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$
\begin{align*}
x_1 & - 5x_3 = 1 \\
x_2 & + x_3 = 4 \\
0 & = 0
\end{align*}
$$

(4)

The variables $x_1$ and $x_2$ corresponding to pivot columns in the matrix are called basic variables. The other variable, $x_3$, is called a free variable.

Whenever a system is consistent, as in (4), the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In (4), solve the first equation for $x_1$ and the second for $x_2$. (Ignore the third equation; it offers no restriction on the variables.)

$$
\begin{align*}
x_1 & = 1 + 5x_3 \\
x_2 & = 4 - x_3
\end{align*}
$$

(5)

The statement “$x_3$ is free” means that you are free to choose any value for $x_3$. Once that is done, the formulas in (5) determine the values for $x_1$ and $x_2$. For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$. Each different choice of $x_3$ determines a (different) solution of the system, and every solution of the system is determined by a choice of $x_3$.

EXAMPLE 4  Find the general solution of the linear system whose augmented matrix has been reduced to

$$
\begin{bmatrix}
1 & 6 & 2 & -5 & -2 & -4 \\
0 & 0 & 2 & -8 & -1 & 3 \\
0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
$$

SOLUTION The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables. The row reduction is completed next. The symbol $\sim$ before a matrix indicates that the matrix is row equivalent to the preceding matrix.

$$
\begin{align*}
\begin{bmatrix}
1 & 6 & 2 & -5 & -2 & -4 \\
0 & 0 & 2 & -8 & -1 & 3 \\
0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix} & \sim 
\begin{bmatrix}
1 & 6 & 2 & -5 & 0 & 10 \\
0 & 0 & 2 & -8 & 0 & 10 \\
0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix} \\
\sim & 
\begin{bmatrix}
1 & 6 & 2 & -5 & 0 & 10 \\
0 & 0 & 1 & -4 & 0 & 5 \\
0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix} \\
& \sim 
\begin{bmatrix}
1 & 6 & 0 & 3 & 0 & 0 \\
0 & 0 & 1 & -4 & 0 & 5 \\
0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
\end{align*}
$$

\footnote{Some texts use the term leading variables because they correspond to the columns containing leading entries.}
There are five variables because the augmented matrix has six columns. The associated system now is

\[
\begin{align*}
x_1 + 6x_2 + 3x_4 &= 0 \\
x_3 - 4x_4 &= 5 \\
x_5 &= 7
\end{align*}
\]  

(6)

The pivot columns of the matrix are 1, 3, and 5, so the basic variables are \(x_1, x_3, \) and \(x_5\). The remaining variables, \(x_2\) and \(x_4\), must be free. Solve for the basic variables to obtain the general solution:

\[
\begin{align*}
x_1 &= -6x_2 - 3x_4 \\
x_2 &\text{ is free} \\
x_3 &= 5 + 4x_4 \\
x_4 &\text{ is free} \\
x_5 &= 7
\end{align*}
\]  

(7)

Note that the value of \(x_3\) is already fixed by the third equation in system (6).

\[\blacksquare\]

**Parametric Descriptions of Solution Sets**

The descriptions in (5) and (7) are *parametric descriptions* of solution sets in which the free variables act as parameters. *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. For instance, in system (4), we may add 5 times equation 2 to equation 1 and obtain the equivalent system

\[
\begin{align*}
x_1 + 5x_2 &= 21 \\
x_2 + x_3 &= 4
\end{align*}
\]

We could treat \(x_2\) as a parameter and solve for \(x_1\) and \(x_3\) in terms of \(x_2\), and we would have an accurate description of the solution set. However, to be consistent, we make the (arbitrary) convention of always using the free variables as the parameters for describing a solution set. (The answer section at the end of the text also reflects this convention.)

Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no* parametric representation.

**Back-Substitution**

Consider the following system, whose augmented matrix is in echelon form but is *not* in reduced echelon form:

\[
\begin{align*}
x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\
x_2 - 3x_3 + 3x_4 + x_5 &= -5 \\
x_4 - x_5 &= 4
\end{align*}
\]

A computer program would solve this system by back-substitution, rather than by computing the reduced echelon form. That is, the program would solve equation 3 for \(x_4\) in terms of \(x_5\) and substitute the expression for \(x_4\) into equation 2, solve equation 2 for \(x_2\), and then substitute the expressions for \(x_2\) and \(x_4\) into equation 1 and solve for \(x_1\).

Our matrix format for the backward phase of row reduction, which produces the reduced echelon form, has the same number of arithmetic operations as back-substitution. But the discipline of the matrix format substantially reduces the likelihood of errors.
during hand computations. The best strategy is to use only the reduced echelon form to solve a system! The Study Guide that accompanies this text offers several helpful suggestions for performing row operations accurately and rapidly.

**NUMERICAL NOTE**

In general, the forward phase of row reduction takes much longer than the backward phase. An algorithm for solving a system is usually measured in flops (or floating point operations). A flop is one arithmetic operation (+, −, *, /) on two real floating point numbers. For an \( n \times (n + 1) \) matrix, the reduction to echelon form can take \( 2n^3/3 + n^2/2 - 7n/6 \) flops (which is approximately \( 2n^3/3 \) flops when \( n \) is moderately large—say, \( n \geq 30 \)). In contrast, further reduction to reduced echelon form needs at most \( n^2 \) flops.

Existence and Uniqueness Questions

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions posed in Section 1.1.

**EXAMPLE 5** Determine the existence and uniqueness of the solutions to the system

\[
\begin{align*}
3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15
\end{align*}
\]

**SOLUTION** The augmented matrix of this system was row reduced in Example 3 to

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

(8)

The basic variables are \( x_1, x_2, \) and \( x_5; \) the free variables are \( x_3 \) and \( x_4. \) There is no equation such as \( 0 = 1 \) that would indicate an inconsistent system, so we could use back-substitution to find a solution. But the existence of a solution is already clear in (8). Also, the solution is not unique because there are free variables. Each different choice of \( x_3 \) and \( x_4 \) determines a different solution. Thus the system has infinitely many solutions.

When a system is in echelon form and contains no equation of the form \( 0 = b, \) with \( b \) nonzero, every nonzero equation contains a basic variable with a nonzero coefficient. Either the basic variables are completely determined (with no free variables) or at least one of the basic variables may be expressed in terms of one or more free variables. In the former case, there is a unique solution; in the latter case, there are infinitely many solutions (one for each choice of values for the free variables).

These remarks justify the following theorem.

---

\(^3\)Traditionally, a flop was only a multiplication or division, because addition and subtraction took much less time and could be ignored. The definition of flop given here is preferred now, as a result of advances in computer architecture. See Golub and Van Loan, *Matrix Computations*, 2nd ed. (Baltimore: The Johns Hopkins Press, 1989), pp. 19–20.
THEOREM 2

Existence and Uniqueness Theorem
A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form
\[
\begin{bmatrix}
0 & \cdots & 0 & b
\end{bmatrix}
\]
with \( b \) nonzero

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM
1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

PRACTICE PROBLEMS

1. Find the general solution of the linear system whose augmented matrix is
\[
\begin{bmatrix}
1 & -3 & -5 & 0 \\
0 & 1 & 1 & 3
\end{bmatrix}
\]

2. Find the general solution of the system
\[
\begin{align*}
x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\
-2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\
3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2
\end{align*}
\]

1.2 EXERCISES

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]  
   b. \[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  
   c. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  
   d. \[
\begin{bmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 3 & 3 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}
\]
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2. a. \[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 2 & 4 \\
0 & 0 & 4 & 8 \\
\end{bmatrix}
\]  
b. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. \[
\begin{bmatrix}
1 & 2 & 4 & 8 \\
2 & 4 & 6 & 8 \\
3 & 6 & 9 & 12 \\
\end{bmatrix}
\]  4. \[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
2 & 4 & 5 & 4 \\
4 & 5 & 4 & 2 \\
\end{bmatrix}
\]

5. Describe the possible echelon forms of a nonzero \(2 \times 2\) matrix. Use the symbols \(\text{\textbullet, \star, \ast}\), and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero \(3 \times 2\) matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. \[
\begin{bmatrix}
1 & 3 & 4 & 7 \\
3 & 9 & 7 & 6 \\
\end{bmatrix}
\]  8. \[
\begin{bmatrix}
1 & -3 & 0 & -5 \\
-3 & 7 & 0 & 9 \\
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
0 & 1 & -2 & 3 \\
1 & -3 & 4 & -6 \\
\end{bmatrix}
\]  10. \[
\begin{bmatrix}
1 & -2 & -1 & 4 \\
-2 & 4 & -5 & 6 \\
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
3 & -2 & 4 & 0 \\
9 & -6 & 12 & 0 \\
6 & -4 & 8 & 0 \\
\end{bmatrix}
\]  12. \[
\begin{bmatrix}
1 & 0 & -9 & 0 & 4 \\
0 & 1 & 3 & 0 & -1 \\
0 & 0 & 0 & 1 & -7 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 & -3 & 0 & -1 & 0 & -2 \\
0 & 1 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 1 & 9 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
1 & 0 & -5 & 0 & -8 & 3 \\
0 & 1 & 4 & -1 & 0 & 6 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Exercises 15 and 16 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

15. a. \[
\begin{bmatrix}
\ast & \ast & \ast \\
0 & \ast & \ast \\
0 & 0 & 0 \\
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
\ast & \ast & \ast & \ast \\
0 & 0 & \ast & \ast \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

16. a. \[
\begin{bmatrix}
\ast & \ast & \ast \\
0 & \ast & \ast \\
0 & 0 & \ast \\
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
\ast & \ast & \ast & \ast & \ast \\
0 & 0 & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast \\
\end{bmatrix}
\]

In Exercises 17 and 18, determine the value(s) of \(h\) such that the matrix is the augmented matrix of a consistent linear system.

17. \[
\begin{bmatrix}
1 & -1 & 4 \\
-2 & 3 & h \\
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
1 & -3 & 1 \\
-2 & h & 6 \\
-2 & 4 & -5 \\
\end{bmatrix}
\]

In Exercises 19 and 20, choose \(h\) and \(k\) such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

19. \[
\begin{align*}
x_1 + hx_2 &= 2 \\
4x_1 + 8x_2 &= k \\
\end{align*}
\]

20. \[
\begin{align*}
x_1 - 3x_2 &= 1 \\
2x_1 + hx_2 &= k \\
\end{align*}
\]

In Exercises 21 and 22, mark each statement True or False. Justify each answer.\(^4\)

21. a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

b. The row reduction algorithm applies only to augmented matrices for a linear system.

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

d. Finding a parametric description of the solution set of a linear system is the same as solving the system.

e. If one row in an echelon form of an augmented matrix is \([0 \ 0 \ 0 \ 5 \ 0]\), then the associated linear system is inconsistent.

22. a. The reduced echelon form of a matrix is unique.

b. If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.

c. The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.

d. A general solution of a system is an explicit description of all solutions of the system.

e. Whenever a system has free variables, the solution set contains many solutions.

23. Suppose the coefficient matrix of a linear system of four equations in four variables has a pivot in each column. Explain why the system has a unique solution.

24. Suppose a system of linear equations has a \(3 \times 5\) augmented matrix whose fifth column is not a pivot column. Is the system consistent? Why (or why not)?

\(^4\) True/false questions of this type will appear in many sections. Methods for justifying your answers were described before Exercises 23 and 24 in Section 1.1.
25. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

26. Suppose a 3 x 5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

27. Restate the last sentence in Theorem 2 using the concept of pivot columns: “If a linear system is consistent, then the solution is unique if and only if ___________.”

28. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

29. A system of linear equations with fewer equations than unknowns is sometimes called an underdetermined system. Can such a system have a unique solution? Explain.

30. Give an example of an inconsistent underdetermined system of two equations in three unknowns.

31. A system of linear equations with more equations than unknowns is sometimes called an overdetermined system. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.

32. Suppose an n x (n + 1) matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when n = 20? When n = 200?

Suppose experimental data are represented by a set of points in the plane. An interpolating polynomial for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

WEB

33. Find the interpolating polynomial \( p(t) = a_0 + a_1t + a_2t^2 \) for the data (1, 6), (2, 15), (3, 28). That is, find \( a_0, a_1, \) and \( a_2 \) such that
\[
\begin{align*}
  a_0 + a_1(1) + a_2(1)^2 &= 6 \\
  a_0 + a_1(2) + a_2(2)^2 &= 15 \\
  a_0 + a_1(3) + a_2(3)^2 &= 28
\end{align*}
\]

34. [M] In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

<table>
<thead>
<tr>
<th>Velocity (ft/sec)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force (100 lb)</td>
<td>0</td>
<td>290</td>
<td>14.8</td>
<td>39.6</td>
<td>74.3</td>
<td>119</td>
</tr>
</tbody>
</table>

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use \( p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 \). What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)

Exercise 33 is a partially completed problem. Let the polynomial \( p(t) = a_0 + a_1t + a_2t^2 \) be the interpolating polynomial for the data. Since \( p(1) = 6 \), \( p(2) = 15 \), \( p(3) = 28 \), we have
\[
\begin{align*}
  a_0 + a_1(1) + a_2(1)^2 &= 6 \\
  a_0 + a_1(2) + a_2(2)^2 &= 15 \\
  a_0 + a_1(3) + a_2(3)^2 &= 28
\end{align*}
\]

The basic variables are \( x_1 \) and \( x_2 \), and the general solution is
\[
\begin{align*}
  x_1 &= 9 + 2x_3 \\
  x_2 &= 3 - x_3 \\
  x_3 & \text{ is free}
\end{align*}
\]

Note: It is essential that the general solution describe each variable, with any parameters clearly identified. The following statement does not describe the solution:
\[
\begin{align*}
  x_1 &= 9 + 2x_3 \\
  x_2 &= 3 - x_3 \\
  x_3 &= 3 - x_2 & \text{Incorrect solution}
\end{align*}
\]

This description implies that \( x_2 \) and \( x_3 \) are both free, which certainly is not the case.
2. Row reduce the system’s augmented matrix:

\[
\begin{bmatrix}
1 & -2 & -1 & 3 & 0 \\
-2 & 4 & 5 & -5 & 3 \\
3 & -6 & -6 & 8 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -2 & -1 & 3 & 0 \\
0 & 0 & 3 & 1 & 3 \\
0 & 0 & -3 & -1 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -2 & -1 & 3 & 0 \\
0 & 0 & 3 & 1 & 3 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}
\]

This echelon matrix shows that the system is *inconsistent*, because its rightmost column is a pivot column; the third row corresponds to the equation \(0 = 5\). There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

### 1.3 Vector Equations

Important properties of linear systems can be described with the concept and notation of vectors. This section connects equations involving vectors to ordinary systems of equations. The term *vector* appears in a variety of mathematical and physical contexts, which we will discuss in Chapter 4, “Vector Spaces.” Until then, *vector* will mean an *ordered list of numbers*. This simple idea enables us to get to interesting and important applications as quickly as possible.

**Vectors in \(\mathbb{R}^2\)**

A matrix with only one column is called a *column vector*, or simply a *vector*. Examples of vectors with two entries are

\[
\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\]

where \(w_1\) and \(w_2\) are any real numbers. The set of all vectors with two entries is denoted by \(\mathbb{R}^2\) (read “r-two”). The \(\mathbb{R}\) stands for the real numbers that appear as entries in the vectors, and the exponent 2 indicates that each vector contains two entries.\(^1\)

Two vectors in \(\mathbb{R}^2\) are *equal* if and only if their corresponding entries are equal. Thus \(\begin{bmatrix} 4 \\ 7 \end{bmatrix}\) and \(\begin{bmatrix} 7 \\ 4 \end{bmatrix}\) are *not* equal, because vectors in \(\mathbb{R}^2\) are *ordered pairs* of real numbers.

Given two vectors \(\mathbf{u}\) and \(\mathbf{v}\) in \(\mathbb{R}^2\), their *sum* is the vector \(\mathbf{u} + \mathbf{v}\) obtained by adding corresponding entries of \(\mathbf{u}\) and \(\mathbf{v}\). For example,

\[
\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 + 2 \\ -2 + 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
\]

Given a vector \(\mathbf{u}\) and a real number \(c\), the *scalar multiple* of \(\mathbf{u}\) by \(c\) is the vector \(c\mathbf{u}\) obtained by multiplying each entry in \(\mathbf{u}\) by \(c\). For instance,

\[
\text{if} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad c = 5, \quad \text{then} \quad c\mathbf{u} = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}
\]

\(^1\) Most of the text concerns vectors and matrices that have only real entries. However, all definitions and theorems in Chapters 1–5, and in most of the rest of the text, remain valid if the entries are complex numbers. Complex vectors and matrices arise naturally, for example, in electrical engineering and physics.
The number $c$ in $cu$ is called a **scalar**; it is written in lightface type to distinguish it from the boldface vector $u$.

The operations of scalar multiplication and vector addition can be combined, as in the following example.

**EXAMPLE 1** Given $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4u$, $-3v$, and $4u + (-3)v$.

**SOLUTION**

$$4u = \begin{bmatrix} 4 \\ -8 \end{bmatrix}, \quad (-3)v = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$

and

$$4u + (-3)v = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

Sometimes, for convenience (and also to save space), this text may write a column vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the form $(3, -1)$. In this case, the parentheses and the comma distinguish the vector $(3, -1)$ from the $1 \times 2$ row matrix $\begin{bmatrix} 3 & -1 \end{bmatrix}$, written with brackets and no comma. Thus

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 3 & -1 \end{bmatrix}$$

because the matrices have different shapes, even though they have the same entries.

**Geometric Descriptions of $\mathbb{R}^2$**

Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point $(a, b)$ with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$. So we may regard $\mathbb{R}^2$ as the set of all points in the plane. See Fig. 1.

![Figure 1: Vectors as points.](image1)

![Figure 2: Vectors with arrows.](image2)

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is often aided by including an arrow (directed line segment) from the origin $(0, 0)$ to the point $(3, -1)$, as in Fig. 2. In this case, the individual points along the arrow itself have no special significance.\(^2\)

The sum of two vectors has a useful geometric representation. The following rule can be verified by analytic geometry.

\(^2\)In physics, arrows can represent forces and usually are free to move about in space. This interpretation of vectors will be discussed in Section 4.1.
The set of all multi
ples of
ual multi
ples of
ue for Addition
If \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^2 \) are represented as points in the plane, then \( \mathbf{u} + \mathbf{v} \) corresponds to the fourth vertex of the parallelogram whose other vertices are \( \mathbf{u}, 0, \) and \( \mathbf{v} \). See Fig. 3.

![Figure 3](image)

**FIGURE 3** The parallelogram rule.

**EXAMPLE 2** The vectors \( \mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix} \), and \( \mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \) are displayed in Fig. 4.

![Figure 4](image)

**EXAMPLE 3** Let \( \mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \). Display the vectors \( \mathbf{u}, 2\mathbf{u}, \) and \( -\frac{2}{3}\mathbf{u} \) on a graph.

**SOLUTION** See Fig. 5, where \( \mathbf{u}, 2\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \), and \( -\frac{2}{3}\mathbf{u} = \begin{bmatrix} -2 \\ 2/3 \end{bmatrix} \) are displayed. The arrow for \( 2\mathbf{u} \) is twice as long as the arrow for \( \mathbf{u} \), and the arrows point in the same direction. The arrow for \( -\frac{2}{3}\mathbf{u} \) is two-thirds the length of the arrow for \( \mathbf{u} \), and the arrows point in opposite directions. In general, the length of the arrow for \( c\mathbf{u} \) is \( |c| \) times the
length of the arrow for \( \mathbf{u} \). [Recall that the length of the line segment from \((0, 0)\) to \((a, b)\) is \( \sqrt{a^2 + b^2} \). We shall discuss this further in Chapter 6.]

**Vectors in \( \mathbb{R}^3 \)**

Vectors in \( \mathbb{R}^3 \) are \( 3 \times 1 \) column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity. The vectors \( \mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) and \( 2\mathbf{a} \) are displayed in Fig. 6.

**Vectors in \( \mathbb{R}^n \)**

If \( n \) is a positive integer, \( \mathbb{R}^n \) (read “r-n”) denotes the collection of all lists (or ordered \( n \)-tuples) of \( n \) real numbers, usually written as \( n \times 1 \) column matrices, such as

\[
\mathbf{u} = \begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_n
\end{bmatrix}
\]

The vector whose entries are all zero is called the **zero vector** and is denoted by \( \mathbf{0} \). (The number of entries in \( \mathbf{0} \) will be clear from the context.)

Equality of vectors in \( \mathbb{R}^n \) and the operations of scalar multiplication and vector addition in \( \mathbb{R}^n \) are defined entry by entry just as in \( \mathbb{R}^2 \). These operations on vectors have the following properties, which can be verified directly from the corresponding properties for real numbers. See Practice Problem 1 and Exercises 33 and 34 at the end of this section.

**Algebraic Properties of \( \mathbb{R}^n \)**

For all \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) in \( \mathbb{R}^n \) and all scalars \( c \) and \( d \):

(i) \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)  \hspace{1cm} (v) \( c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \)

(ii) \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \)  \hspace{1cm} (vi) \( (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} \)

(iii) \( \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \) \hspace{1cm} (vii) \( c(d\mathbf{u}) = (cd)\mathbf{u} \)

(iv) \( \mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0} \), \hspace{1cm} (viii) \( 1\mathbf{u} = \mathbf{u} \)

where \( -\mathbf{u} \) denotes \((-1)\mathbf{u}\)

For simplicity of notation, a vector such as \( \mathbf{u} + (-1)\mathbf{v} \) is often written as \( \mathbf{u} - \mathbf{v} \). Figure 7 shows \( \mathbf{u} - \mathbf{v} \) as the sum of \( \mathbf{u} \) and \(-\mathbf{v}\).

**Linear Combinations**

Given vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \) in \( \mathbb{R}^n \) and given scalars \( c_1, c_2, \ldots, c_p \), the vector \( \mathbf{y} \) defined by

\[
\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p
\]

is called a **linear combination** of \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) with **weights** \( c_1, \ldots, c_p \). Property (ii) above permits us to omit parentheses when forming such a linear combination. The
weights in a linear combination can be any real numbers, including zero. For example, some linear combinations of vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are

\[
\sqrt{3} \mathbf{v}_1 + \mathbf{v}_2, \quad \frac{1}{2} \mathbf{v}_1 \ (= \frac{1}{2} \mathbf{v}_1 + 0 \mathbf{v}_2), \quad \text{and} \quad 0 \ (= 0 \mathbf{v}_1 + 0 \mathbf{v}_2)
\]

**EXAMPLE 4** Figure 8 identifies selected linear combinations of \( \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). (Note that sets of parallel grid lines are drawn through integer multiples of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).) Estimate the linear combinations of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) that generate the vectors \( \mathbf{u} \) and \( \mathbf{w} \).

**SOLUTION** The parallelogram rule shows that \( \mathbf{u} \) is the sum of \( 3 \mathbf{v}_1 \) and \( -2 \mathbf{v}_2 \); that is,

\[
\mathbf{u} = 3 \mathbf{v}_1 - 2 \mathbf{v}_2
\]

This expression for \( \mathbf{u} \) can be interpreted as instructions for traveling from the origin to \( \mathbf{u} \) along two straight paths. First, travel \( 3 \) units in the \( \mathbf{v}_1 \) direction to \( 3 \mathbf{v}_1 \), and then travel \( -2 \) units in the \( \mathbf{v}_2 \) direction (parallel to the line through \( \mathbf{v}_2 \) and \( \mathbf{0} \)). Next, although the vector \( \mathbf{w} \) is not on a grid line, \( \mathbf{w} \) appears to be about halfway between two pairs of grid lines, at the vertex of a parallelogram determined by \( (5/2) \mathbf{v}_1 \) and \( (-1/2) \mathbf{v}_2 \). (See Fig. 9.) Thus a reasonable estimate for \( \mathbf{w} \) is

\[
\mathbf{w} = \frac{5}{2} \mathbf{v}_1 - \frac{1}{2} \mathbf{v}_2
\]

The next example connects a problem about linear combinations to the fundamental existence question studied in Sections 1.1 and 1.2.

**EXAMPLE 5** Let \( \mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} \), \( \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \), and \( \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \). Determine whether \( \mathbf{b} \) can be generated (or written) as a linear combination of \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \). That is, determine whether weights \( x_1 \) and \( x_2 \) exist such that

\[
x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 = \mathbf{b}
\]

If vector equation (1) has a solution, find it.

**SOLUTION** Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

\[
x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}
\]

\[
\uparrow \quad \uparrow \quad \uparrow
\]

\( \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b} \)
which is the same as
\[
\begin{bmatrix}
  x_1 \\
  -2x_1 \\
  -5x_1
\end{bmatrix}
+ \begin{bmatrix}
  2x_2 \\
  5x_2 \\
  6x_2
\end{bmatrix} = \begin{bmatrix}
  7 \\
  4 \\
  -3
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
  x_1 + 2x_2 \\
  -2x_1 + 5x_2 \\
  -5x_1 + 6x_2
\end{bmatrix} = \begin{bmatrix}
  7 \\
  4 \\
  -3
\end{bmatrix}
\tag{2}
\]

The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, \(x_1\) and \(x_2\) make the vector equation (1) true if and only if \(x_1\) and \(x_2\) satisfy the system
\[
\begin{align*}
  x_1 + 2x_2 &= 7 \\
  -2x_1 + 5x_2 &= 4 \\
  -5x_1 + 6x_2 &= -3
\end{align*}
\tag{3}
\]

To solve this system, row reduce the augmented matrix of the system as follows:\footnote{The symbol \(\sim\) between matrices denotes row equivalence (Section 1.2).}
\[
\begin{bmatrix}
  1 & 2 & 7 \\
  -2 & 5 & 4 \\
  -5 & 6 & -3
\end{bmatrix} \sim \begin{bmatrix}
  1 & 2 & 7 \\
  0 & 9 & 18 \\
  0 & 16 & 32
\end{bmatrix} \sim \begin{bmatrix}
  1 & 0 & 3 \\
  0 & 1 & 2 \\
  0 & 0 & 0
\end{bmatrix}
\]
The solution of (3) is \(x_1 = 3\) and \(x_2 = 2\). Hence \(\mathbf{b}\) is a linear combination of \(\mathbf{a}_1\) and \(\mathbf{a}_2\), with weights \(x_1 = 3\) and \(x_2 = 2\). That is,
\[
3 \begin{bmatrix}
  1 \\
  -2 \\
  -5
\end{bmatrix} + 2 \begin{bmatrix}
  2 \\
  5 \\
  6
\end{bmatrix} = \begin{bmatrix}
  7 \\
  4 \\
  -3
\end{bmatrix}
\]

Observe in Example 5 that the original vectors \(\mathbf{a}_1\), \(\mathbf{a}_2\), and \(\mathbf{b}\) are the columns of the augmented matrix that we row reduced:
\[
\begin{bmatrix}
  1 & 2 & 7 \\
  -2 & 5 & 4 \\
  -5 & 6 & -3
\end{bmatrix}
\]

For brevity, write this matrix in a way that identifies its columns—namely,
\[
\begin{bmatrix}
  \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b}
\end{bmatrix}
\tag{4}
\]

It is clear how to write this augmented matrix immediately from vector equation (1), without going through the intermediate steps of Example 5. Take the vectors in the order in which they appear in (1) and put them into the columns of a matrix as in (4).

The discussion above is easily modified to establish the following fundamental fact.

\[\text{A vector equation } x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \]

has the same solution set as the linear system whose augmented matrix is
\[
\begin{bmatrix}
  \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b}
\end{bmatrix}
\tag{5}
\]

In particular, \(\mathbf{b}\) can be generated by a linear combination of \(\mathbf{a}_1, \ldots, \mathbf{a}_n\) if and only if there exists a solution to the linear system corresponding to the matrix (5).
One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) of vectors.

**Definition**

If \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) are in \( \mathbb{R}^n \), then the set of all linear combinations of \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) is denoted by \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) and is called the **subset of \( \mathbb{R}^n \) spanned (or generated) by \( \mathbf{v}_1, \ldots, \mathbf{v}_p \)**. That is, \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) is the collection of all vectors that can be written in the form

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p
\]

with \( c_1, \ldots, c_p \) scalars.

Asking whether a vector \( \mathbf{b} \) is in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) amounts to asking whether the vector equation

\[
x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_p \mathbf{v}_p = \mathbf{b}
\]

has a solution, or, equivalently, asking whether the linear system with augmented matrix

\[
\begin{bmatrix}
\mathbf{v}_1 & \cdots & \mathbf{v}_p & \mathbf{b}
\end{bmatrix}
\]

has a solution.

Note that \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) contains every scalar multiple of \( \mathbf{v}_1 \) (for example), since \( c \mathbf{v}_1 = c \mathbf{v}_1 + 0 \mathbf{v}_2 + \cdots + 0 \mathbf{v}_p \). In particular, the zero vector must be in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \).

**A Geometric Description of \( \text{Span} \{ \mathbf{v} \} \) and \( \text{Span} \{ \mathbf{u}, \mathbf{v} \} \)**

Let \( \mathbf{v} \) be a nonzero vector in \( \mathbb{R}^3 \). Then \( \text{Span} \{ \mathbf{v} \} \) is the set of all scalar multiples of \( \mathbf{v} \), which is the set of points on the line in \( \mathbb{R}^3 \) through \( \mathbf{v} \) and \( \mathbf{0} \). See Fig. 10.

If \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors in \( \mathbb{R}^3 \), with \( \mathbf{v} \) not a multiple of \( \mathbf{u} \), then \( \text{Span} \{ \mathbf{u}, \mathbf{v} \} \) is the plane in \( \mathbb{R}^3 \) that contains \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{0} \). In particular, \( \text{Span} \{ \mathbf{u}, \mathbf{v} \} \) contains the line in \( \mathbb{R}^3 \) through \( \mathbf{u} \) and \( \mathbf{0} \) and the line through \( \mathbf{v} \) and \( \mathbf{0} \). See Fig. 11.

**Example 6**

Let \( \mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \), \( \mathbf{a}_2 = \begin{bmatrix} -5 \\ -13 \\ -3 \end{bmatrix} \), and \( \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \). Then \( \text{Span} \{ \mathbf{a}_1, \mathbf{a}_2 \} \) is a plane through the origin in \( \mathbb{R}^3 \). Is \( \mathbf{b} \) in that plane?
**Solution** Does the equation \( x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 = \mathbf{b} \) have a solution? To answer this, row reduce the augmented matrix \( \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix} \):

\[
\begin{bmatrix}
  1 & 5 & -3 \\
-2 & -13 & 8 \\
 3 & -3 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 5 & -3 \\
0 & -3 & 2 \\
0 & -18 & 10
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 5 & -3 \\
0 & -3 & 2 \\
0 & 0 & -2
\end{bmatrix}
\]

The third equation is \( 0 = -2 \), which shows that the system has no solution. The vector equation \( x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 = \mathbf{b} \) has no solution, and so \( \mathbf{b} \) is not in \( \text{Span} \{ \mathbf{a}_1, \mathbf{a}_2 \} \).

**Linear Combinations in Applications**

The final example shows how scalar multiples and linear combinations can arise when a quantity such as “cost” is broken down into several categories. The basic principle for the example concerns the cost of producing several units of an item when the cost per unit is known:

\[
\begin{bmatrix}
\text{number} \\
\text{of units}
\end{bmatrix}
\cdot
\begin{bmatrix}
\text{cost} \\
\text{per unit}
\end{bmatrix}
= \begin{bmatrix}
\text{total cost}
\end{bmatrix}
\]

**Example 7** A company manufactures two products. For $1.00 worth of product B, the company spends $.45 on materials, $.25 on labor, and $.15 on overhead. For $1.00 worth of product C, the company spends $.40 on materials, $.30 on labor, and $.15 on overhead. Let

\[
\mathbf{b} = \begin{bmatrix}
.45 \\
.25 \\
.15
\end{bmatrix}
\text{ and }\mathbf{c} = \begin{bmatrix}
.40 \\
.30 \\
.15
\end{bmatrix}
\]

Then \( \mathbf{b} \) and \( \mathbf{c} \) represent the “costs per dollar of income” for the two products.

a. What economic interpretation can be given to the vector \( 100\mathbf{b} \)?

b. Suppose the company wishes to manufacture \( x_1 \) dollars worth of product B and \( x_2 \) dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor, and overhead).

**Solution**

a. Compute

\[
100\mathbf{b} = 100 \begin{bmatrix}
.45 \\
.25 \\
.15
\end{bmatrix} = \begin{bmatrix}
45 \\
25 \\
15
\end{bmatrix}
\]

The vector \( 100\mathbf{b} \) lists the various costs for producing $100 worth of product B—namely, $45 for materials, $25 for labor, and $15 for overhead.

b. The costs of manufacturing \( x_1 \) dollars worth of B are given by the vector \( x_1 \mathbf{b} \), and the costs of manufacturing \( x_2 \) dollars worth of C are given by \( x_2 \mathbf{c} \). Hence the total costs for both products are given by the vector \( x_1 \mathbf{b} + x_2 \mathbf{c} \).

**Practice Problems**

1. Prove that \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \) for any \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \).

2. For what value(s) of \( h \) will \( \mathbf{y} \) be in \( \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) if

\[
\mathbf{v}_1 = \begin{bmatrix}
1 \\
-1 \\
-2
\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}
5 \\
-4 \\
-7
\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}
-3 \\
1 \\
0
\end{bmatrix}, \quad \text{and } \mathbf{y} = \begin{bmatrix}
-4 \\
3 \\
h
\end{bmatrix}
\]
1.3 EXERCISES

In Exercises 1 and 2, compute \( u + v \) and \( u - 2v \).

1. \( u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \)

2. \( u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \)

In Exercises 3 and 4, display the following vectors using arrows on an xy-graph: \( u, v, -v, -2v, u + v, u - v, \) and \( u - 2v \). Notice that \( u - v \) is the vertex of a parallelogram whose other vertices are \( u, 0, \) and \( -v \).

3. \( u \) and \( v \) as in Exercise 1

4. \( u \) and \( v \) as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. \( x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix} \)

6. \( x_1 \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of \( u \) and \( v \). Is every vector in \( \mathbb{R}^2 \) a linear combination of \( u \) and \( v \)?

7. Vectors \( a, b, c, \) and \( d \)

8. Vectors \( w, x, y, \) and \( z \)

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. \( x_2 + 5x_3 = 0 \)

10. \( 3x_1 - 2x_2 + 4x_3 = 3 \\
    4x_1 + 6x_2 - x_3 = 0 \\
    -2x_1 - 7x_2 + 5x_3 = 1 \\
    -x_1 + 3x_2 - 8x_3 = 0 \\
    5x_1 + 4x_2 - 3x_3 = 2 \)

In Exercises 11 and 12, determine if \( b \) is a linear combination of \( a_1, a_2, \) and \( a_3 \).

11. \( a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \)

12. \( a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -6 \\ 7 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 5 \end{bmatrix} \)

In Exercises 13 and 14, determine if \( b \) is a linear combination of the vectors formed from the columns of the matrix \( A \).

13. \( A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix} \)

14. \( A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \)

15. Let \( a_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -8 \\ 2 \end{bmatrix}, \) and \( b = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \). For what value(s) of \( h \) is \( b \) in the plane spanned by \( a_1 \) and \( a_2 \)?

16. Let \( v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, \) and \( y = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix} \). For what value(s) of \( h \) is \( y \) in the plane generated by \( v_1 \) and \( v_2 \)?

In Exercises 17 and 18, list five vectors in \( \text{Span} \{v_1, v_2\} \). For each vector, show the weights on \( v_1 \) and \( v_2 \) used to generate the vector and list the three entries of the vector. Do not make a sketch.

17. \( v_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \)

18. \( v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \)

19. Give a geometric description of \( \text{Span} \{v_1, v_2\} \) for the vectors \( v_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix} \).

20. Give a geometric description of \( \text{Span} \{v_1, v_2\} \) for the vectors in Exercise 18.

21. Let \( u = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \) and \( v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Show that \( \begin{bmatrix} h \\ k \end{bmatrix} \) is in \( \text{Span} \{u, v\} \) for all \( h \) and \( k \).

22. Construct a \( 3 \times 3 \) matrix \( A \), with nonzero entries, and a vector \( b \) in \( \mathbb{R}^3 \) such that \( b \) is not in the set spanned by the columns of \( A \).

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector \( \begin{bmatrix} -4 \\ 3 \end{bmatrix} \) is \( [-4, 3] \).

   b. The points in the plane corresponding to \( \begin{bmatrix} -2 \\ 5 \end{bmatrix} \) and \( \begin{bmatrix} -5 \\ 2 \end{bmatrix} \) lie on a line through the origin.

   c. An example of a linear combination of vectors \( v_1 \) and \( v_2 \) is the vector \( \frac{1}{2}v_1 \).
d. The solution set of the linear system whose augmented matrix is \([a_1, a_2, a_3, b]\) is the same as the solution set of the equation \(x_1a_1 + x_2a_2 + x_3a_3 = b\).

e. The set \(\text{Span}\{\mathbf{u}, \mathbf{v}\}\) is always visualized as a plane through the origin.

24. a. When \(\mathbf{u}\) and \(\mathbf{v}\) are nonzero vectors, \(\text{Span}\{\mathbf{u}, \mathbf{v}\}\) contains only the line through \(\mathbf{u}\) and the origin, and the line through \(\mathbf{v}\) and the origin.

b. Any list of five real numbers is a vector in \(\mathbb{R}^5\).

c. Asking whether the linear system corresponding to an augmented matrix \([a_1, a_2, a_3, b]\) has a solution amounts to asking whether \(b\) is in \(\text{Span}\{a_1, a_2, a_3\}\).

d. The vector \(\mathbf{v}\) results when a vector \(\mathbf{u} - \mathbf{v}\) is added to the vector \(\mathbf{v}\).

e. The weights \(c_1, \ldots, c_p\) in a linear combination \(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p\) cannot all be zero.

25. Let \(A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}\) and \(b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}\). Denote the columns of \(A\) by \(a_1, a_2, a_3\), and let \(W = \text{Span}\{a_1, a_2, a_3\}\).

a. Is \(b\) in \(\{a_1, a_2, a_3\}\)? How many vectors are in \(\{a_1, a_2, a_3\}\)?

b. Is \(b\) in \(W\)? How many vectors are in \(W\)?

c. Show that \(a_3\) is in \(W\). [Hint: Row operations are unnecessary.]

26. Let \(A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}\), let \(b = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}\), and let \(W\) be the set of all linear combinations of the columns of \(A\).

a. Is \(b\) in \(W\)?

b. Show that the second column of \(A\) is in \(W\).

27. A mining company has two mines. One day’s operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kilograms of silver, while one day’s operation at mine #2 produces ore that contains 40 metric tons of copper and 380 kilograms of silver. Let \(\mathbf{v}_1 = \begin{bmatrix} 30 \\ 600 \end{bmatrix}\) and \(\mathbf{v}_2 = \begin{bmatrix} 40 \\ 380 \end{bmatrix}\). Then \(\mathbf{v}_1\) and \(\mathbf{v}_2\) represent the “output per day” of mine #1 and mine #2, respectively.

a. What physical interpretation can be given to the vector \(5\mathbf{v}_1\)?

b. Suppose the company operates mine #1 for \(x_1\) days and mine #2 for \(x_2\) days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 240 tons of copper and 2824 kilograms of silver. Do not solve the equation.

c. [M] Solve the equation in (b).

28. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

a. How much heat does the steam plant produce when it burns \(x_1\) tons of A and \(x_2\) tons of B?

b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns \(x_1\) tons of A and \(x_2\) tons of B.

c. [M] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

29. Let \(\mathbf{v}_1, \ldots, \mathbf{v}_k\) be points in \(\mathbb{R}^3\) and suppose that for \(j = 1, \ldots, k\) an object with mass \(m_j\) is located at point \(\mathbf{v}_j\). Physicists call such objects point masses. The total mass of the system of point masses is

\[ m = m_1 + \cdots + m_k \]

The center of gravity (or center of mass) of the system is

\[ \mathbf{v} = \frac{1}{m} [m_1 \mathbf{v}_1 + \cdots + m_k \mathbf{v}_k] \]

Compute the center of gravity of the system consisting of the following point masses (see the figure):

<table>
<thead>
<tr>
<th>Point</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbf{v}_1 = (2, -2, 4))</td>
<td>4 g</td>
</tr>
<tr>
<td>(\mathbf{v}_2 = (-4, 2, 3))</td>
<td>2 g</td>
</tr>
<tr>
<td>(\mathbf{v}_3 = (4, 0, -2))</td>
<td>3 g</td>
</tr>
<tr>
<td>(\mathbf{v}_4 = (1, -6, 0))</td>
<td>5 g</td>
</tr>
</tbody>
</table>

30. Let \(\mathbf{v}\) be the center of mass of a system of point masses located at \(\mathbf{v}_1, \ldots, \mathbf{v}_k\) as in Exercise 29. Is \(\mathbf{v}\) in \(\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}\)? Explain.
34. **CHAPTER 1  Linear Equations in Linear Algebra**

31. A thin triangular plate of uniform density and thickness has vertices at $v_1 = (0, 1)$, $v_2 = (8, 1)$, and $v_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.

![Diagram of a triangular plate with vertices labeled $v_1$, $v_2$, and $v_3$]

a. Find the $(x, y)$-coordinates of the center of mass of the plate. This “balance point” of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.

b. Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to $(2, 2)$. [Hint: Let $w_1$, $w_2$, and $w_3$ denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]

32. Consider the vectors $v_1$, $v_2$, $v_3$, and $b$ in $\mathbb{R}^2$, shown in the figure. Does the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ have a solution? Is the solution unique? Use the figure to explain your answers.

![Diagram of vectors $v_1$, $v_2$, $v_3$, and $b$]

33. Use the vectors $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, and $w = (w_1, \ldots, w_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

a. $(u + v) + w = u + (v + w)$

b. $c(u + v) = cu + cv$ for each scalar $c$

34. Use the vector $u = (u_1, \ldots, u_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

a. $u + (-u) = (-u) + u = 0$

b. $c(du) = (cd)u$ for all scalars $c$ and $d$

---

**SOLUTIONS TO PRACTICE PROBLEMS**

1. Take arbitrary vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, and compute

   $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$  \hspace{1cm} \text{Definition of vector addition}

   $= (u_1 + u_1, \ldots, v_n + u_n)$  \hspace{1cm} \text{Commutativity of addition in } \mathbb{R}

   $= \mathbf{v} + \mathbf{u}$  \hspace{1cm} \text{Definition of vector addition}

2. The vector $y$ belongs to Span $\{v_1, v_2, v_3\}$ if and only if there exist scalars $x_1, x_2, x_3$ such that

   \[
   \begin{bmatrix}
   x_1 & 1 \\
   -1 & -2 \\
   h & 3
   \end{bmatrix}
   \begin{bmatrix}
   -4 \\
   3 \\
   h
   \end{bmatrix}
   \]

   This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

   \[
   \begin{bmatrix}
   1 & 5 & -3 & -4 \\
   -1 & -4 & 1 & 3 \\
   -2 & 7 & 0 & h
   \end{bmatrix}
   =
   \begin{bmatrix}
   1 & 5 & -3 & -4 \\
   0 & 1 & -2 & -1 \\
   0 & 3 & -6 & h - 5
   \end{bmatrix}
   \]

   The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So $y$ is in Span $\{v_1, v_2, v_3\}$ if and only if $h = 5$.

   \text{Remember:} \hspace{1cm} \text{The presence of a free variable in a system does not guarantee that the system is consistent.}

---

1.4 **THE MATRIX EQUATION $Ax = b$**

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.
DEFINITION

If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of $A$ and $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of the columns of $A$ using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$
A\mathbf{x} = \begin{bmatrix}
\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n
\end{bmatrix} \begin{bmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\vdots \\
\mathbf{x}_n
\end{bmatrix} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n
$$

Note that $A\mathbf{x}$ is defined only if the number of columns of $A$ equals the number of entries in $\mathbf{x}$.

EXAMPLE 1

a. \[ \begin{bmatrix}
1 & 2 & -1 \\
0 & -5 & 3
\end{bmatrix} \begin{bmatrix}
4 \\
3
\end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \]

b. \[ \begin{bmatrix}
2 & -3 \\
8 & 0 \\
-5 & 2
\end{bmatrix} \begin{bmatrix}
4 \\
7
\end{bmatrix} = 4 \begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix} \]

EXAMPLE 2 For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix $A$ and place the weights $3, -5,$ and 7 into a vector $\mathbf{x}$. That is,

$$
3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3
\end{bmatrix} \begin{bmatrix}
3 \\
-5 \\
7
\end{bmatrix} = A\mathbf{x}
$$

Section 1.3 showed how to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 4 \\
-5x_2 + 3x_3 &= 1
\end{align*}
\]

is equivalent to

\[ x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{2} \]

As in Example 2, the linear combination on the left side is a matrix times a vector, so that (2) becomes

\[ \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{3} \]

Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a matrix equation, to distinguish it from a vector equation such as is shown in (2).

Notice how the matrix in (3) is just the matrix of coefficients of the system (1). Similar calculations show that any system of linear equations, or any vector equation such as (2), can be written as an equivalent matrix equation in the form $A\mathbf{x} = \mathbf{b}$. This simple observation will be used repeatedly throughout the text.

Here is the formal result.
THEOREM 3

If $A$ is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if $b$ is in $\mathbb{R}^m$, the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

Theorem 3 provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient. In any case, the matrix equation (4), the vector equation (5), and the system of equations are all solved in the same way—by row reducing the augmented matrix (6). Other methods of solution will be discussed later.

Existence of Solutions

The definition of $Ax$ leads directly to the following useful fact.

The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

Section 1.3 considered the existence question, “Is $b$ in $\text{Span}\{a_1, \ldots, a_n\}$?” Equivalently, “Is $Ax = b$ consistent?” A harder existence problem is to determine whether the equation $Ax = b$ is consistent for all possible $b$.

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible $b_1, b_2, b_3$?

SOLUTION Row reduce the augmented matrix for $Ax = b$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{7}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{7}b_2 + b_3$. The equation $Ax = b$ is not consistent for every $b$ because some choices of $b$ can make $b_1 - \frac{1}{7}b_2 + b_3$ nonzero.
The reduced matrix in Example 3 provides a description of all \( \mathbf{b} \) for which the equation \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is consistent: The entries in \( \mathbf{b} \) must satisfy

\[
b_1 - \frac{1}{2} b_2 + b_3 = 0
\]

This is the equation of a plane through the origin in \( \mathbb{R}^3 \). The plane is the set of all linear combinations of the three columns of \( \mathbf{A} \). See Fig. 1.

The equation \( \mathbf{A}\mathbf{x} = \mathbf{b} \) in Example 3 fails to be consistent for all \( \mathbf{b} \) because the echelon form of \( \mathbf{A} \) has a row of zeros. If \( \mathbf{A} \) had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as \([0 \ 0 \ 1 \ 1]\).

In the next theorem, the sentence “The columns of \( \mathbf{A} \) span \( \mathbb{R}^m \)” means that every \( \mathbf{b} \) in \( \mathbb{R}^m \) is a linear combination of the columns of \( \mathbf{A} \). In general, a set of vectors \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) in \( \mathbb{R}^m \) spans (or generates) \( \mathbb{R}^m \) if every vector in \( \mathbb{R}^m \) is a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_p \)—that is, if \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} = \mathbb{R}^m \).

**Theorem 4**  

Let \( \mathbf{A} \) be an \( m \times n \) matrix. Then the following statements are logically equivalent. That is, for a particular \( \mathbf{A} \), either they are all true statements or they are all false.

a. For each \( \mathbf{b} \) in \( \mathbb{R}^m \), the equation \( \mathbf{A}\mathbf{x} = \mathbf{b} \) has a solution.

b. Each \( \mathbf{b} \) in \( \mathbb{R}^m \) is a linear combination of the columns of \( \mathbf{A} \).

c. The columns of \( \mathbf{A} \) span \( \mathbb{R}^m \).

d. \( \mathbf{A} \) has a pivot position in every row.

Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of \( \mathbf{A}\mathbf{x} \) and what it means for a set of vectors to span \( \mathbb{R}^m \). The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

**Warning:** Theorem 4 is about a coefficient matrix, not an augmented matrix. If an augmented matrix \( [ \mathbf{A} \ \mathbf{b} ] \) has a pivot position in every row, then the equation \( \mathbf{A}\mathbf{x} = \mathbf{b} \) may or may not be consistent.

**Computation of \( \mathbf{A}\mathbf{x} \)**

The calculations in Example 1 were based on the definition of the product of a matrix \( \mathbf{A} \) and a vector \( \mathbf{x} \). The following simple example will lead to a more efficient method for calculating the entries in \( \mathbf{A}\mathbf{x} \) when working problems by hand.

**Example 4**  

Compute \( \mathbf{A}\mathbf{x} \), where \( \mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \) and \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \).
SOLUTION From the definition,

\[
\begin{bmatrix}
2 & 3 & 4 \\
-1 & 5 & -3 \\
6 & -2 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}
\]

\[
= \begin{bmatrix}
2x_1 \\
-x_1 \\
6x_1
\end{bmatrix} + \begin{bmatrix}
3x_2 \\
5x_2 \\
-2x_2
\end{bmatrix} + \begin{bmatrix}
4x_3 \\
-3x_3 \\
8x_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2x_1 + 3x_2 + 4x_3 \\
-x_1 + 5x_2 - 3x_3 \\
6x_1 - 2x_2 + 8x_3
\end{bmatrix}
\] (7)

The first entry in the product \(Ax\) is a sum of products (sometimes called a dot product), using the first row of \(A\) and the entries in \(x\). That is,

\[
\begin{bmatrix}
2 & 3 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
2x_1 + 3x_2 + 4x_3
\end{bmatrix}
\]

This matrix shows how to compute the first entry in \(Ax\) directly, without writing down all the calculations shown in (7). Similarly, the second entry in \(Ax\) can be calculated at once by multiplying the entries in the second row of \(A\) by the corresponding entries in \(x\) and then summing the resulting products:

\[
\begin{bmatrix}
-1 & 5 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
-x_1 + 5x_2 - 3x_3
\end{bmatrix}
\]

Likewise, the third entry in \(Ax\) can be calculated from the third row of \(A\) and the entries in \(x\).

Row–Vector Rule for Computing \(Ax\)

If the product \(Ax\) is defined, then the \(i\)th entry in \(Ax\) is the sum of the products of corresponding entries from row \(i\) of \(A\) and from the vector \(x\).

EXAMPLE 5

a. \[
\begin{bmatrix}
1 & 2 & -1 \\
0 & -5 & 3
\end{bmatrix}
\begin{bmatrix}
4 \\
3 \\
7
\end{bmatrix}
= \begin{bmatrix}
1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\
0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7
\end{bmatrix}
= \begin{bmatrix}
3 \\
6
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
2 & -3 \\
8 & 0 \\
-5 & 2
\end{bmatrix}
\begin{bmatrix}
4 \\
7
\end{bmatrix}
= \begin{bmatrix}
2 \cdot 4 + (-3) \cdot 7 \\
8 \cdot 4 + 0 \cdot 7 \\
(-5) \cdot 4 + 2 \cdot 7
\end{bmatrix}
= \begin{bmatrix}
-13 \\
32 \\
6
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
r \\
s \\
t
\end{bmatrix}
= \begin{bmatrix}
1 \cdot r + 0 \cdot s + 0 \cdot t \\
0 \cdot r + 1 \cdot s + 0 \cdot t \\
0 \cdot r + 0 \cdot s + 1 \cdot t
\end{bmatrix}
= \begin{bmatrix}
r \\
s \\
t
\end{bmatrix}
\]

By definition, the matrix in Example 5(c) with 1’s on the diagonal and 0’s elsewhere is called an identity matrix and is denoted by \(I\). The calculation in part (c) shows that \(I \mathbf{x} = \mathbf{x}\) for every \(\mathbf{x}\) in \(\mathbb{R}^3\). There is an analogous \(n \times n\) identity matrix, sometimes written as \(I_n\). As in part (c), \(I_n \mathbf{x} = \mathbf{x}\) for every \(\mathbf{x}\) in \(\mathbb{R}^n\).
Properties of the Matrix–Vector Product \( Ax \)

The facts in the next theorem are important and will be used throughout the text. The proof relies on the definition of \( Ax \) and the algebraic properties of \( \mathbb{R}^n \).

**Theorem 5**

If \( A \) is an \( m \times n \) matrix, \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in \( \mathbb{R}^n \), and \( c \) is a scalar, then:

a. \( A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} \);

b. \( A(c\mathbf{u}) = c(A\mathbf{u}) \).

**Proof** For simplicity, take \( n = 3 \), \( A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \), and \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \). (The proof of the general case is similar.) For \( i = 1, 2, 3 \), let \( u_i \) and \( v_i \) be the \( i \)th entries in \( \mathbf{u} \) and \( \mathbf{v} \), respectively. To prove statement (a), compute \( A(\mathbf{u} + \mathbf{v}) \) as a linear combination of the columns of \( A \) using the entries in \( \mathbf{u} + \mathbf{v} \) as weights.

\[
A(\mathbf{u} + \mathbf{v}) = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3 = (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) = A\mathbf{u} + A\mathbf{v}
\]

To prove statement (b), compute \( A(c\mathbf{u}) \) as a linear combination of the columns of \( A \) using the entries in \( c\mathbf{u} \) as weights.

\[
A(c\mathbf{u}) = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 = c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) = c(A\mathbf{u})
\]

**Numerical Note**

To optimize a computer algorithm to compute \( Ax \), the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute \( Ax \) as a linear combination of the columns of \( A \). In contrast, if a program is written in the popular language C, which stores matrices by rows, \( A x \) should be computed via the alternative rule that uses the rows of \( A \).

**Proof of Theorem 4** As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix \( A \)) that (a) and (d) are either both true or both false. That will tie all four statements together.

Let \( \mathbf{U} \) be an echelon form of \( A \). Given \( \mathbf{b} \) in \( \mathbb{R}^m \), we can row reduce the augmented matrix \( [ A \quad \mathbf{b} ] \) to an augmented matrix \( [ \mathbf{U} \quad \mathbf{d} ] \) for some \( \mathbf{d} \) in \( \mathbb{R}^m \):

\[
[A \quad b] \sim \cdots \sim [U \quad d]
\]
If statement (d) is true, then each row of $U$ contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true. If (d) is false, the last row of $U$ is all zeros. Let $\mathbf{d}$ be any vector with a 1 in its last entry. Then $[U\ \mathbf{d}]$ represents an inconsistent system. Since row operations are reversible, $[U\ \mathbf{d}]$ can be transformed into the form $[A\ \mathbf{b}]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false.

**PRACTICE PROBLEMS**

1. Let $A = \begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}$, $p = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, and $b = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$. It can be shown that $p$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of $A$.

2. Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -1 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.

1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix}$ $\begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 1 & 2 \\ -2 & -3 \\ -3 & 1 \\ 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -3 & -2 \\ 3 & 2 & 1 \\ 8 & -5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} -3 \\ -3 \\ 5 \\ 11 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ 49 \\ 11 \end{bmatrix}$

7. $x_1 \begin{bmatrix} 4 \\ -7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -7 \end{bmatrix}$

8. $z_1 \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

9. $5x_1 + x_2 - 3x_3 = 8$ $10. 4x_1 - x_2 = 8$

$2x_2 + 4x_3 = 0$ $5x_1 + 3x_2 = 2$

$3x_1 - x_2 = 1$

Given $A$ and $\mathbf{b}$ in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

11. $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

13. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of $A$? (See the figure.) Why or why not?

14. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of $A$? Why or why not?
15. Let \( A = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix} \) and \( b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \). Show that the equation \( Ax = b \) does not have a solution for all possible \( b \), and describe the set of all \( b \) for which \( Ax = b \) does have a solution.

16. Repeat the requests from Exercise 15 with
   \( A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix} \), and \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \).

Exercises 17–20 refer to the matrices \( A \) and \( B \) below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

\[
A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}.
\]

17. How many rows of \( A \) contain a pivot position? Does the equation \( Ax = b \) have a solution for each \( b \) in \( \mathbb{R}^4 \)?

18. Can every vector in \( \mathbb{R}^4 \) be written as a linear combination of the columns of the matrix \( B \) above? Do the columns of \( B \) span \( \mathbb{R}^4 \)?

19. Can each vector in \( \mathbb{R}^4 \) be written as a linear combination of the columns of the matrix \( A \) above? Do the columns of \( A \) span \( \mathbb{R}^4 \)?

20. Do the columns of \( B \) span \( \mathbb{R}^4 \)? Does the equation \( Bx = y \) have a solution for each \( y \) in \( \mathbb{R}^4 \)?

21. Let \( v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \). Does \( \{v_1, v_2, v_3\} \) span \( \mathbb{R}^3 \)? Why or why not?

22. Let \( v_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix} \). Does \( \{v_1, v_2, v_3\} \) span \( \mathbb{R}^3 \)? Why or why not?

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. The equation \( Ax = b \) is referred to as a vector equation.
   b. A vector \( b \) is a linear combination of the columns of a matrix \( A \) if and only if the equation \( Ax = b \) has at least one solution.
   c. The equation \( Ax = b \) is consistent if the augmented matrix \( [ A \ | \ b ] \) has a pivot position in every row.
   d. The first entry in the product \( Ax \) is a sum of products.
   e. If the columns of an \( m \times n \) matrix \( A \) span \( \mathbb{R}^m \), then the equation \( Ax = b \) is consistent for each \( b \) in \( \mathbb{R}^m \).
   f. If \( A \) is an \( m \times n \) matrix and if the equation \( Ax = b \) is inconsistent for some \( b \) in \( \mathbb{R}^m \), then \( A \) cannot have a pivot position in every row.

24. a. Every matrix equation \( Ax = b \) corresponds to a vector equation with the same solution set.
   b. If the equation \( Ax = b \) is consistent, then \( b \) is in the set spanned by the columns of \( A \).
   c. Any linear combination of vectors can always be written in the form \( Ax \) for a suitable matrix \( A \) and vector \( x \).
   d. If the coefficient matrix \( A \) has a pivot position in every row, then the equation \( Ax = b \) is inconsistent.
   e. The solution set of a linear system whose augmented matrix is \( [ a_1 \ a_2 \ a_3 \ b ] \) is the same as the solution set of \( Ax = b \), if \( A = [ a_1 \ a_2 \ a_3 ] \).
   f. If \( A \) is an \( m \times n \) matrix whose columns do not span \( \mathbb{R}^m \), then the equation \( Ax = b \) is consistent for every \( b \) in \( \mathbb{R}^n \).

25. Note that \( \begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -10 \end{bmatrix} \). Use this fact (and no row operations) to find scalars \( c_1, c_2, c_3 \) such that \( \begin{bmatrix} -7 \\ -10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \end{bmatrix} \).

26. Let \( u = \begin{bmatrix} 7 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \) and \( w = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \). It can be shown that \( 2u - 3v - w = 0 \). Use this fact (and no row operations) to find \( x_1 \) and \( x_2 \) that satisfy the equation \( \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \).

27. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols \( v_1, v_2, \ldots \) for the vectors and \( c_1, c_2, \ldots \) for scalars. Define what each symbol represents, using the data given in the matrix equation.

\[
\begin{bmatrix} -3 & 5 & 4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \end{bmatrix}
\]

28. Let \( q_1, q_2, q_3, \) and \( v \) represent vectors in \( \mathbb{R}^5 \), and let \( x_1, x_2, \) and \( x_3 \) denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use. \( x_1q_1 + x_2q_2 + x_3q_3 = v \)

29. Construct a \( 3 \times 3 \) matrix, not in echelon form, whose columns span \( \mathbb{R}^3 \). Show that the matrix you construct has the desired property.

30. Construct a \( 3 \times 3 \) matrix, not in echelon form, whose columns do not span \( \mathbb{R}^3 \). Show that the matrix you construct has the desired property.

31. Let \( A \) be a \( 3 \times 2 \) matrix. Explain why the equation \( Ax = b \) cannot be consistent for all \( b \) in \( \mathbb{R}^3 \). Generalize your argument to the case of an arbitrary \( A \) with more rows than columns.
32. Could a set of three vectors in \( \mathbb{R}^4 \) span all of \( \mathbb{R}^4 \)? Explain. What about \( n \) vectors in \( \mathbb{R}^m \) when \( n \) is less than \( m \)?

33. Suppose \( A \) is a \( 4 \times 3 \) matrix and \( \mathbf{b} \) is a vector in \( \mathbb{R}^4 \) with the property that \( A\mathbf{x} = \mathbf{b} \) has a unique solution. What can you say about the reduced echelon form of \( A \)? Justify your answer.

34. Let \( A \) be a \( 3 \times 4 \) matrix, let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be vectors in \( \mathbb{R}^3 \), and let \( \mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 \). Suppose \( \mathbf{v}_1 = A\mathbf{u}_1 \) and \( \mathbf{v}_2 = A\mathbf{u}_2 \) for some vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) in \( \mathbb{R}^4 \). What fact allows you to conclude that the system \( A\mathbf{x} = \mathbf{w} \) is consistent? (Note: \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) denote vectors, not scalar entries in vectors.)

35. Let \( A \) be a \( 5 \times 3 \) matrix, let \( \mathbf{y} \) be a vector in \( \mathbb{R}^3 \), and let \( \mathbf{z} \) be a vector in \( \mathbb{R}^5 \). Suppose \( A\mathbf{y} = \mathbf{z} \). What fact allows you to conclude that the system \( A\mathbf{x} = 5\mathbf{z} \) is consistent?

36. Suppose \( A \) is a \( 4 \times 4 \) matrix and \( \mathbf{b} \) is a vector in \( \mathbb{R}^4 \) with the property that \( A\mathbf{x} = \mathbf{b} \) has a unique solution. Explain why the columns of \( A \) must span \( \mathbb{R}^4 \).

[M] In Exercises 37–40, determine if the columns of the matrix span \( \mathbb{R}^4 \).

**SOLUTIONS TO PRACTICE PROBLEMS**

1. The matrix equation

\[
\begin{bmatrix}
1 & 5 & -2 & 0 \\
-3 & 1 & 9 & -5 \\
4 & -8 & -1 & 7
\end{bmatrix}
\begin{bmatrix}
3 \\
-2 \\
0 \\
-4
\end{bmatrix} =
\begin{bmatrix}
-7 \\
9 \\
0
\end{bmatrix}
\]

is equivalent to the vector equation

\[
3
\begin{bmatrix}
1 \\
-3 \\
4
\end{bmatrix} - 2
\begin{bmatrix}
5 \\
1 \\
-8
\end{bmatrix} + 0
\begin{bmatrix}
-2 \\
9 \\
-1
\end{bmatrix} - 4
\begin{bmatrix}
0 \\
-5 \\
7
\end{bmatrix} =
\begin{bmatrix}
-7 \\
9 \\
0
\end{bmatrix}
\]

which expresses \( \mathbf{b} \) as a linear combination of the columns of \( A \).

2. \( \mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \)

\[
A(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 + 20 \\ 3 + 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}
\]

\[A\mathbf{u} + Av = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} =
\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 19 \\ -4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix} \]

1.5 SOLUTION SETS OF LINEAR SYSTEMS

Solution sets of linear systems are important objects of study in linear algebra. They will appear later in several different contexts. This section uses vector notation to give explicit and geometric descriptions of such solution sets.

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form \( Ax = 0 \), where \( A \) is an \( m \times n \) matrix and \( 0 \) is the zero vector in \( \mathbb{R}^m \). Such a system \( Ax = 0 \) **always** has at least one solution, namely, \( x = 0 \) (the zero vector in \( \mathbb{R}^n \)). This zero solution is usually called the **trivial solution**. For a given equation \( Ax = 0 \), the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector \( x \) that satisfies \( Ax = 0 \). The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

The homogeneous equation \( Ax = 0 \) has a nontrivial solution if and only if the equation has at least one free variable.

**EXAMPLE 1** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

\[
\begin{align*}
3x_1 + 5x_2 - 4x_3 &= 0 \\
-3x_1 - 2x_2 + 4x_3 &= 0 \\
6x_1 + x_2 - 8x_3 &= 0
\end{align*}
\]

**SOLUTION** Let \( A \) be the matrix of coefficients of the system and row reduce the augmented matrix \([ A \ 0]\) to echelon form:

\[
\begin{bmatrix}
3 & 5 & -4 & 0 \\
-3 & -2 & 4 & 0 \\
6 & 1 & -8 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 5 & -4 & 0 \\
0 & 3 & 0 & 0 \\
0 & -9 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
3 & 5 & -4 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since \( x_3 \) is a free variable, \( Ax = 0 \) has nontrivial solutions (one for each choice of \( x_3 \)). To describe the solution set, continue the row reduction of \([ A \ 0]\) to **reduced** echelon form:

\[
\begin{bmatrix}
1 & 0 & -\frac{4}{3} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x_1 & = & -\frac{4}{3}x_3 \\
x_2 & = & 0 \\
x_3 & = & 0
\end{bmatrix}
\]

Solve for the basic variables \( x_1 \) and \( x_2 \) and obtain \( x_1 = \frac{4}{3}x_3, x_2 = 0 \), with \( x_3 \) free. As a vector, the general solution of \( Ax = 0 \) has the form

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = x_3 \begin{bmatrix}
\frac{4}{3} \\
0 \\
1
\end{bmatrix} = x_3v,
\]

where \( v = \begin{bmatrix}
\frac{4}{3} \\
0 \\
1
\end{bmatrix} \). Here \( x_3 \) is factored out of the expression for the general solution vector. This shows that every solution of \( Ax = 0 \) in this case is a scalar multiple of \( v \). The trivial solution is obtained by choosing \( x_3 = 0 \). Geometrically, the solution set is a line through \( 0 \) in \( \mathbb{R}^3 \). See Fig. 1.
Notice that a nontrivial solution \( \mathbf{x} \) can have some zero entries so long as not all of its entries are zero.

**EXAMPLE 2** A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous “system”

\[
10x_1 - 3x_2 - 2x_3 = 0
\]

**SOLUTION** There is no need for matrix notation. Solve for the basic variable \( x_1 \) in terms of the free variables. The general solution is \( x_1 = .3x_2 + .2x_3 \), with \( x_2 \) and \( x_3 \) free. As a vector, the general solution is

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ 1 \end{bmatrix} \quad (\text{with } x_2, x_3 \text{ free})
\]

This calculation shows that every solution of (1) is a linear combination of the vectors \( \mathbf{u} \) and \( \mathbf{v} \), shown in (2). That is, the solution set is \( \text{Span} \{ \mathbf{u}, \mathbf{v} \} \). Since neither \( \mathbf{u} \) nor \( \mathbf{v} \) is a scalar multiple of the other, the solution set is a plane through the origin. See Fig. 2. ■

Examples 1 and 2, along with the exercises, illustrate the fact that the solution set of a homogeneous equation \( A\mathbf{x} = \mathbf{0} \) can always be expressed explicitly as \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) for suitable vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_p \). If the only solution is the zero vector, then the solution set is \( \text{Span} \{ \mathbf{0} \} \). If the equation \( A\mathbf{x} = \mathbf{0} \) has only one free variable, the solution set is a line through the origin, as in Fig. 1. A plane through the origin, as in Fig. 2, provides a good mental image for the solution set of \( A\mathbf{x} = \mathbf{0} \) when there are two or more free variables. Note, however, that a similar figure can be used to visualize \( \text{Span} \{ \mathbf{u}, \mathbf{v} \} \) even when \( \mathbf{u} \) and \( \mathbf{v} \) do not arise as solutions of \( A\mathbf{x} = \mathbf{0} \). See Fig. 11 in Section 1.3.

**Parametric Vector Form**

The original equation (1) for the plane in Example 2 is an *implicit* description of the plane. Solving this equation amounts to finding an *explicit* description of the plane as the set spanned by \( \mathbf{u} \) and \( \mathbf{v} \). Equation (2) is called a **parametric vector equation** of the plane. Sometimes such an equation is written as

\[
\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})
\]

to emphasize that the parameters vary over all real numbers. In Example 1, the equation \( \mathbf{x} = x_3\mathbf{v} \) (with \( x_3 \) free), or \( \mathbf{x} = t\mathbf{v} \) (with \( t \) in \( \mathbb{R} \)), is a parametric vector equation of a line. Whenever a solution set is described explicitly with vectors as in Examples 1 and 2, we say that the solution is in **parametric vector form**.

**Solutions of Nonhomogeneous Systems**

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.
EXAMPLE 3  Describe all solutions of $Ax = b$, where

$$
A = \begin{bmatrix}
  3 & 5 & -4 \\
-3 & -2 & 4 \\
 6 & 1 & -8 
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
 7 \\
-1 \\
-4
\end{bmatrix}
$$

SOLUTION  Here $A$ is the matrix of coefficients from Example 1. Row operations on $[ A \ b ]$ produce

$$
\begin{bmatrix}
  3 & 5 & -4 & 7 \\
-3 & -2 & 4 & -1 \\
 6 & 1 & -8 & -4 
\end{bmatrix} \sim
\begin{bmatrix}
  1 & 0 & -4 & 1/3 \\
 0 & 1 & 2 & 0 \\
 0 & 0 & 0 & 0
\end{bmatrix}, \quad x_1 - \frac{4}{3} x_3 = -1, \quad x_2 = 2, \quad 0 = 0
$$

Thus $x_1 = -1 + \frac{4}{3} x_3$, $x_2 = 2$, and $x_3$ is free. As a vector, the general solution of $Ax = b$ has the form

$$
\mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-1 + \frac{4}{3} x_3 \\
2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-1 \\
2 \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{4}{3} x_3 \\
0 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-1 \\
2 \\
0
\end{bmatrix} + x_3 \begin{bmatrix}
\frac{4}{3} \\
0 \\
1
\end{bmatrix}
$$

The equation $\mathbf{x} = \mathbf{p} + x_3 \mathbf{v}$, or, writing $t$ as a general parameter,

$$
\mathbf{x} = \mathbf{p} + t \mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{3}
$$

describes the solution set of $Ax = b$ in parametric vector form. Recall from Example 1 that the solution set of $Ax = 0$ has the parametric vector equation

$$
\mathbf{x} = t \mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{4}
$$

[with the same $\mathbf{v}$ that appears in (3)]. Thus the solutions of $Ax = b$ are obtained by adding the vector $\mathbf{p}$ to the solutions of $Ax = 0$. The vector $\mathbf{p}$ itself is just one particular solution of $Ax = b$ [corresponding to $t = 0$ in (3)].

To describe the solution set of $Ax = b$ geometrically, we can think of vector addition as a translation. Given $\mathbf{v}$ and $\mathbf{p}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to move $\mathbf{v}$ in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is translated by $\mathbf{p}$ to $\mathbf{v} + \mathbf{p}$. See Fig. 3. If each point on a line $L$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector $\mathbf{p}$, the result is a line parallel to $L$. See Fig. 4.

Suppose $L$ is the line through $\mathbf{0}$ and $\mathbf{v}$, described by equation (4). Adding $\mathbf{p}$ to each point on $L$ produces the translated line described by equation (3). Note that $\mathbf{p}$ is on the line in equation (3). We call (3) the equation of the line through $\mathbf{p}$ parallel to $\mathbf{v}$. Thus the solution set of $Ax = b$ is a line through $\mathbf{p}$ parallel to the solution set of $Ax = 0$. Figure 5 illustrates this case.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example3.png}
\caption{Parallel solution sets of $Ax = b$ and $Ax = 0$.}
\end{figure}

The relation between the solution sets of $Ax = b$ and $Ax = 0$ shown in Fig. 5 generalizes to any consistent equation $Ax = b$, although the solution set will be larger than a line when there are several free variables. The following theorem gives the precise statement. See Exercise 25 for a proof.
THEOREM 6
Suppose the equation \( Ax = b \) is consistent for some given \( b \), and let \( p \) be a solution. Then the solution set of \( Ax = b \) is the set of all vectors of the form \( w = p + v_h \), where \( v_h \) is any solution of the homogeneous equation \( Ax = 0 \).

Theorem 6 says that if \( Ax = b \) has a solution, then the solution set is obtained by translating the solution set of \( Ax = 0 \), using any particular solution \( p \) of \( Ax = b \) for the translation. Figure 6 illustrates the case in which there are two free variables. Even when \( n > 3 \), our mental image of the solution set of a consistent system \( Ax = b \) (with \( b \neq 0 \)) is either a single nonzero point or a line or plane not passing through the origin.

![Figure 6](image-url)

**FIGURE 6** Parallel solution sets of \( Ax = b \) and \( Ax = 0 \).

**Warning:** Theorem 6 and Fig. 6 apply only to an equation \( Ax = b \) that has at least one nonzero solution \( p \). When \( Ax = b \) has no solution, the solution set is empty.

The following algorithm outlines the calculations shown in Examples 1, 2, and 3.

<table>
<thead>
<tr>
<th>WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Row reduce the augmented matrix to reduced echelon form.</td>
</tr>
<tr>
<td>2. Express each basic variable in terms of any free variables appearing in an equation.</td>
</tr>
<tr>
<td>3. Write a typical solution ( x ) as a vector whose entries depend on the free variables, if any.</td>
</tr>
<tr>
<td>4. Decompose ( x ) into a linear combination of vectors (with numeric entries) using the free variables as parameters.</td>
</tr>
</tbody>
</table>

**PRACTICE PROBLEMS**

1. Each of the following equations determines a plane in \( \mathbb{R}^3 \). Do the two planes intersect? If so, describe their intersection.

   \[
   x_1 + 4x_2 - 5x_3 = 0 \\
   2x_1 - x_2 + 8x_3 = 9
   \]

2. Write the general solution of \( 10x_1 - 3x_2 - 2x_3 = 7 \) in parametric vector form, and relate the solution set to the one found in Example 2.
1.5 EXERCISES

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. \[ \begin{align*} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{align*} \]

2. \[ \begin{align*} x_1 - 2x_2 + 3x_3 &= 0 \\ -2x_1 - 3x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 9x_3 &= 0 \end{align*} \]

3. \[ \begin{align*} -3x_1 + 4x_2 - 8x_3 &= 0 \\ -2x_1 + 5x_2 + 4x_3 &= 0 \end{align*} \]

4. \[ \begin{align*} 5x_1 - 3x_2 + 2x_3 &= 0 \\ -3x_1 - 4x_2 + 2x_3 &= 0 \end{align*} \]

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5. \[ \begin{align*} 2x_1 + 2x_2 + 4x_3 &= 0 \\ -4x_1 - 4x_2 - 8x_3 &= 0 \\ -3x_1 - 3x_2 &= 0 \end{align*} \]

6. \[ \begin{align*} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -1x_1 + x_2 &= 0 \end{align*} \]

In Exercises 7–12, describe all solutions of \( Ax = 0 \) in parametric vector form, where \( A \) is row equivalent to the given matrix.

7. \[ \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} \]

8. \[ \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix} \]

9. \[ \begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix} \]

10. \[ \begin{bmatrix} -1 & -4 & 0 & -4 \\ 2 & -8 & 0 & 8 \end{bmatrix} \]

11. \[ \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

12. \[ \begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

13. Suppose the solution set of a certain system of linear equations can be described as \( x_1 = 5 + 4x_3 \), \( x_2 = -2 - 7x_3 \), with \( x_3 \) free. Use vectors to describe this set as a line in \( \mathbb{R}^3 \).

14. Suppose the solution set of a certain system of linear equations can be described as \( x_1 = 5x_4, \ x_2 = 3 - 2x_4, \ x_3 = 2 + 5x_4 \), with \( x_4 \) free. Use vectors to describe this set as a “line” in \( \mathbb{R}^4 \).

15. Describe and compare the solution sets of \( x_1 + 5x_2 - 3x_3 = 0 \) and \( x_1 + 5x_2 - 3x_3 = -2 \).

16. Describe and compare the solution sets of \( x_1 - 2x_2 + 3x_3 = 0 \) and \( x_1 - 2x_2 + 3x_3 = 4 \).

17. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

\[ \begin{align*} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{align*} \]

18. As in Exercise 17, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

\[ \begin{align*} x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 13 \\ -x_1 + x_2 &= -8 \end{align*} \]

In Exercises 19 and 20, find the parametric equation of the line through \( a \) parallel to \( b \).

19. \( a = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \ b = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \)

20. \( a = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \ b = \begin{bmatrix} -7 \\ 6 \end{bmatrix} \)

In Exercises 21 and 22, find a parametric equation of the line \( M \) through \( p \) and \( q \). [Hint: \( M \) is parallel to the vector \( q - p \). See the figure below.]

21. \( p = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \ q = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

22. \( p = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \ q = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \)

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A homogeneous equation is always consistent.
   b. The equation \( Ax = 0 \) gives an explicit description of its solution.
   c. The homogeneous equation \( Ax = 0 \) has the trivial solution if and only if the equation has at least one free variable.
   d. The equation \( x = p + tv \) describes a line through \( v \) parallel to \( p \).
   e. The solution set of \( Ax = b \) is the set of all vectors of the form \( w = p + v_0 \), where \( v_0 \) is any solution of the equation \( Ax = 0 \).

24. a. A homogeneous system of equations can be inconsistent.
   b. If \( x \) is a nontrivial solution of \( Ax = 0 \), then every entry in \( x \) is nonzero.
   c. The effect of adding \( p \) to a vector is to move the vector in a direction parallel to \( p \).
   d. The equation \( Ax = b \) is homogeneous if the zero vector is a solution.
e. If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

25. Prove Theorem 6:
   a. Suppose $p$ is a solution of $Ax = b$, so that $Ap = b$. Let $v_h$ be any solution of the homogeneous equation $Ax = 0$, and let $w = p + v_h$. Show that $w$ is a solution of $Ax = b$.
   b. Let $w$ be any solution of $Ax = b$, define $v_h = w - p$. Show that $v_h$ is a solution of $Ax = 0$. This shows that every solution of $Ax = b$ has the form $w = p + v_h$, with $p$ a particular solution of $Ax = b$ and $v_h$ a solution of $Ax = 0$.

26. Suppose $A$ is the $3 \times 3$ zero matrix (with all zero entries). Describe the solution set of the equation $Ax = 0$.

27. Suppose $Ax = b$ has a solution. Explain why the solution is unique precisely when $Ax = 0$ has only the trivial solution.

In Exercises 28–31, (a) does the equation $Ax = 0$ have a nontrivial solution and (b) does the equation $Ax = b$ have at least one solution for every possible $b$?

28. $A$ is a $3 \times 3$ matrix with three pivot positions.
29. $A$ is a $4 \times 4$ matrix with three pivot positions.
30. $A$ is a $2 \times 5$ matrix with two pivot positions.
31. $A$ is a $3 \times 2$ matrix with two pivot positions.
32. If $b \neq 0$, can the solution set of $Ax = b$ be a plane through the origin? Explain.

33. Construct a $3 \times 3$ nonzero matrix $A$ such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

34. Construct a $3 \times 3$ nonzero matrix $A$ such that the vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

35. Given $A = \begin{bmatrix} -1 & -3 \\ 7 & 21 \\ -2 & -6 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection. [Hint: Think of the equation $Ax = 0$ written as a vector equation.]

36. Given $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection.

37. Construct a $2 \times 2$ matrix $A$ such that the solution set of the equation $Ax = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $b$ in $\mathbb{R}^2$ such that the solution set of $Ax = b$ is not a line in $\mathbb{R}^2$ parallel to the solution set of $Ax = 0$. Why does this not contradict Theorem 6?

38. Let $A$ be an $m \times n$ matrix and let $w$ be a vector in $\mathbb{R}^n$ that satisfies the equation $Ax = 0$. Show that for any scalar $c$, the vector $cw$ also satisfies $Ax = 0$. [That is, show that $A(cw) = 0$.]

39. Let $A$ be an $m \times n$ matrix, and let $v$ and $w$ be vectors in $\mathbb{R}^n$ with the property that $Av = 0$ and $Aw = 0$. Explain why $A(v + w)$ must be the zero vector. Then explain why $A(cv + dw) = 0$ for each pair of scalars $c$ and $d$.

40. Suppose $A$ is a $3 \times 3$ matrix and $b$ is a vector in $\mathbb{R}^3$ such that the equation $Ax = b$ does not have a solution. Does there exist a vector $y$ in $\mathbb{R}^3$ such that the equation $Ax = y$ has a unique solution? Discuss.

---

**SOLUTIONS TO PRACTICE PROBLEMS**

1. Row reduce the augmented matrix:

\[
\begin{bmatrix}
1 & 4 & -5 & 0 \\
2 & -1 & 8 & 9
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 4 & -5 & 0 \\
0 & -9 & 18 & 9
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 3 & 4 \\
0 & 1 & -2 & -1
\end{bmatrix}
\]

Thus $x_1 = 4 - 3x_3, x_2 = -1 + 2x_3$, with $x_3$ free. The general solution in parametric vector form is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
4 - 3x_3 \\
-1 + 2x_3 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
4 \\
-1 \\
0
\end{bmatrix}
+ x_3 \begin{bmatrix}
-3 \\
2 \\
1
\end{bmatrix}
\]

The intersection of the two planes is the line through $p$ in the direction of $v$. 

2. The augmented matrix \[
\begin{bmatrix}
10 & -3 & -2 & 7
\end{bmatrix}
\] is row equivalent to \[
\begin{bmatrix}
1 & -3 & -2 & 7
\end{bmatrix}
\] and the general solution is \( x_1 = .7 + .3x_2 + .2x_3 \), with \( x_2 \) and \( x_3 \) free. That is,
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 + .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} = p + x_2 u + x_3 v
\]
The solution set of the nonhomogeneous equation \( Ax = b \) is the translated plane \( p + \text{Span}\{u, v\} \), which passes through \( p \) and is parallel to the solution set of the homogeneous equation in Example 2.

### 1.6 Applications of Linear Systems

You might expect that a real-life problem involving linear algebra would have only one solution, or perhaps no solution. The purpose of this section is to show how linear systems with many solutions can arise naturally. The applications here come from economics, chemistry, and network flow.

#### A Homogeneous System in Economics

The system of 500 equations in 500 variables, mentioned in this chapter’s introduction, is now known as a Leontief “input–output” (or “production”) model.1 Section 2.6 will examine this model in more detail, when more theory and better notation are available. For now, we look at a simpler “exchange model,” also due to Leontief.

Suppose a nation’s economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries. Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or “exchanged” among the other sectors of the economy. Let the total dollar value of a sector’s output be called the price of that output. Leontief proved the following result.

There exist equilibrium prices that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

The following example shows how to find the equilibrium prices.

#### Example 1

Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1 on page 50, where the entries in a column represent the fractional parts of a sector’s total output.

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

---

Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by \( p_C \), \( p_E \), and \( p_S \), respectively. If possible, find equilibrium prices that make each sector’s income match its expenditures.

**SOLUTION** A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs. For instance, the first row of Table 1 says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output. Since the respective values of the total outputs are \( p_E \) and \( p_S \), Coal must spend \( .4p_E \) dollars for its share of Electric’s output and \( .6p_S \) for its share of Steel’s output. Thus Coal’s total expenses are \( .4p_E + .6p_S \). To make Coal’s income, \( p_C \), equal to its expenses, we want

\[
p_C = .4p_E + .6p_S
\]  

(1)

The second row of the exchange table shows that the Electric sector spends \( .6p_C \) for coal, \( .1p_E \) for electricity, and \( .2p_S \) for steel. Hence the income/expense requirement for Electric is

\[
p_E = .6p_C + .1p_E + .2p_S
\]  

(2)

Finally, the third row of the exchange table leads to the final requirement:

\[
p_S = .4p_C + .5p_E + .2p_S
\]  

(3)

To solve the system of equations (1), (2), and (3), move all the unknowns to the left sides of the equations and combine like terms. [For instance, on the left side of (2), write \( p_E - .1p_E \) as \( .9p_E \).]

\[
p_C - .4p_E - .6p_S = 0
\]

\[
-.6p_C + .9p_E - .2p_S = 0
\]

\[
-.4p_C - .5p_E + .8p_S = 0
\]

Row reduction is next. For simplicity here, decimals are rounded to two places.

\[
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
-.6 & .9 & -.2 & 0 \\
-.4 & -.5 & .8 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
0 & .66 & -.56 & 0 \\
0 & -.66 & .56 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -.4 & -.6 & 0 \\
0 & 0 & -.94 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

**TABLE 1** A Simple Economy

<table>
<thead>
<tr>
<th>Distribution of Output</th>
<th>Purchased by</th>
<th>Coal</th>
<th>Electric</th>
<th>Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coal</td>
<td>.0</td>
<td>.4</td>
<td>.6</td>
<td>Coal</td>
</tr>
<tr>
<td>Electric</td>
<td>.6</td>
<td>.1</td>
<td>.2</td>
<td>Electric</td>
</tr>
<tr>
<td>Steel</td>
<td>.4</td>
<td>.5</td>
<td>.2</td>
<td>Steel</td>
</tr>
</tbody>
</table>
The general solution is $p_C = .94p_S$, $p_E = .85p_S$, and $p_S$ is free. The equilibrium price vector for the economy has the form

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}.$$  

Any (nonnegative) choice for $p_S$ results in a choice of equilibrium prices. For instance, if we take $p_S$ to be 100 (or $100 million), then $p_C = 94$ and $p_E = 85$. The incomes and expenditures of each sector will be equal if the output of Coal is priced at $94 million, that of Electric at $85 million, and that of Steel at $100 million.

### Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane ($C_3H_8$) combines with oxygen ($O_2$) to form carbon dioxide ($CO_2$) and water ($H_2O$), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O$$  \hspace{1cm} (4)

To “balance” this equation, a chemist must find whole numbers $x_1, \ldots, x_4$ such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right (because atoms are neither destroyed nor created in the reaction).

A systematic method for balancing chemical equations is to set up a vector equation that describes the numbers of atoms of each type present in a reaction. Since equation (4) involves three types of atoms (carbon, hydrogen, and oxygen), construct a vector in $\mathbb{R}^3$ for each reactant and product in (4) that lists the numbers of “atoms per molecule,” as follows:

$$\begin{align*}
C_3H_8: & \quad \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, & O_2: & \quad \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, & CO_2: & \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, & H_2O: & \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \hspace{1cm} \text{Carbon} \\
& & & & & & \hspace{1cm} \text{Hydrogen} \\
& & & & & & \hspace{1cm} \text{Oxygen}
\end{align*}$$

To balance equation (4), the coefficients $x_1, \ldots, x_4$ must satisfy

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

To solve, move all the terms to the left (changing the signs in the third and fourth vectors):

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reduction of the augmented matrix for this equation leads to the general solution

$$x_1 = \frac{1}{3}x_4, \quad x_2 = \frac{2}{3}x_4, \quad x_3 = \frac{1}{3}x_4, \quad \text{with } x_4 \text{ free}$$

Since the coefficients in a chemical equation must be integers, take $x_4 = 4$, in which case $x_1 = 1, x_2 = 5$, and $x_3 = 3$. The balanced equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

The equation would also be balanced if, for example, each coefficient were doubled. For most purposes, however, chemists prefer to use a balanced equation whose coefficients are the smallest possible whole numbers.
Network Flow

Systems of linear equations arise naturally when scientists, engineers, or economists study the flow of some quantity through a network. For instance, urban planners and traffic engineers monitor the pattern of traffic flow in a grid of city streets. Electrical engineers calculate current flow through electrical circuits. And economists analyze the distribution of products from manufacturers to consumers through a network of wholesalers and retailers. For many networks, the systems of equations involve hundreds or even thousands of variables and equations.

A network consists of a set of points called junctions, or nodes, with lines or arcs called branches connecting some or all of the junctions. The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction. For example, Fig. 1 shows 30 units flowing into a junction through one branch, with $x_1$ and $x_2$ denoting the flows out of the junction through other branches. Since the flow is "conserved" at each junction, we must have $x_1 + x_2 = 30$. In a similar fashion, the flow at each junction is described by a linear equation. The problem of network analysis is to determine the flow in each branch when partial information (such as the flow into and out of the network) is known.

**EXAMPLE 2** The network in Fig. 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

**SOLUTION** Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in Fig. 2. At each intersection, set the flow in equal to the flow out.

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Flow in</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$300 + 500$</td>
<td>$x_1 + x_2$</td>
</tr>
<tr>
<td>B</td>
<td>$x_2 + x_4$</td>
<td>$300 + x_3$</td>
</tr>
<tr>
<td>C</td>
<td>$100 + 400$</td>
<td>$x_4 + x_5$</td>
</tr>
<tr>
<td>D</td>
<td>$x_1 + x_5$</td>
<td>$600$</td>
</tr>
</tbody>
</table>
Also, the total flow into the network (500 + 300 + 100 + 400) equals the total flow out of the network (300 + \( x_3 \) + 600), which simplifies to \( x_3 = 400 \). Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:

\[
\begin{align*}
    x_1 + x_2 &= 800 \\
    x_2 - x_3 + x_4 &= 300 \\
    x_4 + x_5 &= 500 \\
    x_1 + x_5 &= 600 \\
    x_3 &= 400
\end{align*}
\]

Row reduction of the associated augmented matrix leads to

\[
\begin{align*}
    x_1 + x_5 &= 600 \\
    x_2 - x_5 &= 200 \\
    x_3 &= 400 \\
    x_4 + x_5 &= 500
\end{align*}
\]

The general flow pattern for the network is described by

\[
\begin{align*}
    x_1 &= 600 - x_5 \\
    x_2 &= 200 + x_5 \\
    x_3 &= 400 \\
    x_4 &= 500 - x_5 \\
    x_5 \text{ is free}
\end{align*}
\]

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one-way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, \( x_5 \leq 500 \) because \( x_4 \) cannot be negative. Other constraints on the variables are considered in Practice Problem 2.

**PRACTICE PROBLEMS**

1. Suppose an economy has three sectors: Agriculture, Mining, and Manufacturing. Agriculture sells 5\% of its output to Mining and 30\% to Manufacturing, and retains the rest. Mining sells 20\% of its output to Agriculture and 70\% to Manufacturing, and retains the rest. Manufacturing sells 20\% of its output to Agriculture and 30\% to Mining, and retains the rest. Determine the exchange table for this economy, where the columns describe how the output of each sector is exchanged among the three sectors.

2. Consider the network flow studied in Example 2. Determine the possible range of values of \( x_1 \) and \( x_2 \). [Hint: The example showed that \( x_5 \leq 500 \). What does this imply about \( x_1 \) and \( x_2 \)? Also, use the fact that \( x_5 \geq 0 \).]
1.6 Exercises

1. Suppose an economy has only two sectors: Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector’s income match its expenditures.

2. Find another set of equilibrium prices for the economy in Example 1. Suppose the same economy used Japanese yen instead of dollars to measure the values of the various sectors’ outputs. Would this change the problem in any way? Discuss.

3. Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to Services, and retains the rest. Manufacturing sells 10% of its output to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.
   a. Construct the exchange table for this economy.
   b. Develop a system of equations that leads to prices at which each sector’s income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.
   c. [M] Find a set of equilibrium prices when the price for the Services output is 100 units.

4. Suppose an economy has four sectors: Mining, Lumber, Energy, and Transportation. Mining sells 10% of its output to Lumber, 60% to Energy, and retains the rest. Lumber sells 15% of its output to Mining, 50% to Energy, 20% to Transportation, and retains the rest. Energy sells 20% of its output to Lumber, 15% to Lumber, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to Mining, 10% to Lumber, 50% to Energy, and retains the rest.
   a. Construct the exchange table for this economy.
   b. [M] Find a set of equilibrium prices for the economy.

5. An economy has four sectors: Agriculture, Manufacturing, Services, and Transportation. Agriculture sells 20% of its output to Manufacturing, 30% to Services, 30% to Transportation, and retains the rest. Manufacturing sells 35% of its output to Agriculture, 35% to Services, 20% to Transportation, and retains the rest. Services sells 10% of its output to Agriculture, 20% to Manufacturing, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to Agriculture, 30% to Manufacturing, 20% to Services, and retains the rest.
   a. Construct the exchange table for this economy.
   b. [M] Find a set of equilibrium prices for the economy if the value of Transportation is $10.00 per unit.
   c. The Services sector launches a successful “eat farm fresh” campaign, and increases its share of the output from the Agricultural sector to 40%, whereas the share of Agricultural production going to Manufacturing falls to 10%. Construct the exchange table for this new economy.
   d. [M] Find a set of equilibrium prices for this new economy if the value of Transportation is still $10.00 per unit. What effect has the “eat farm fresh” campaign had on the equilibrium prices for the sectors in this economy?

6. Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide:
   \[ \text{Al}_2\text{O}_3 + 3 \text{C} \rightarrow 2 \text{Al} + 3 \text{CO}_2 \]
   [For each compound, construct a vector that lists the numbers of atoms of aluminum, oxygen, and carbon.]

7. Alka-Seltzer contains sodium bicarbonate \((\text{NaHCO}_3)\) and citric acid \((\text{H}_3\text{C}_6\text{H}_5\text{O}_7)\). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):
   \[ \text{NaHCO}_3 + \text{H}_3\text{C}_6\text{H}_5\text{O}_7 \rightarrow \text{Na}_2\text{C}_6\text{H}_5\text{O}_7 + \text{H}_2\text{O} + \text{CO}_2 \]

8. Limestone, \(\text{CaCO}_3\), neutralizes the acid, \(\text{H}_2\text{O}\), in acid rain by the following unbalanced equation:
   \[ \text{H}_2\text{O} + \text{CaCO}_3 \rightarrow \text{H}_2\text{O} + \text{Ca} + \text{CO}_2 \]

9. Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs). The unbalanced equation is
   \[ \text{B}_2\text{S}_3 + \text{H}_2\text{O} \rightarrow \text{H}_3\text{BO}_3 + \text{H}_2\text{S} \]

10. [M] If possible, use exact arithmetic or a rational format for calculations in balancing the following chemical reaction:
    \[ \text{PhN}_6 + \text{CrMn}_2\text{O}_8 \rightarrow \text{Ph}_2\text{O}_4 + \text{Cr}_2\text{O}_3 + \text{MnO}_2 + \text{NO} \]

11. [M] The chemical reaction below can be used in some industrial processes, such as the production of arsine (\(\text{AsH}_3\)). Use exact arithmetic or a rational format for calculations to balance this equation.
    \[ \text{MnS} + \text{As}_2\text{Cr}_10\text{O}_{35} + \text{H}_2\text{SO}_4 \rightarrow \text{HMnO}_4 + \text{AsH}_3 + \text{Cr}_3\text{O}_12 + \text{H}_2\text{O} \]
12. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for $x_4$?

![Flow Network Diagram](image)

b. Describe the general traffic pattern when the road whose flow is $x_5$ is closed.

c. When $x_5 = 0$, what is the minimum value of $x_4$?

13. a. Find the general flow pattern of the network shown in the figure.

b. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by $x_2$, $x_3$, $x_4$, and $x_5$.

![Flow Network Diagram](image)

14. a. Find the general traffic pattern of the freeway network shown in the figure. (Flow rates are in cars/minute.)

![Freeway Network Diagram](image)

b. Describe the general traffic pattern when the road whose flow is $x_5$ is closed.

c. When $x_5 = 0$, what is the minimum value of $x_4$?

15. Intersections in England are often constructed as one-way “roundabouts,” such as the one shown in the figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for $x_6$.

![Roundabout Diagram](image)

### SOLUTIONS TO PRACTICE PROBLEMS

1. Write the percentages as decimals. Since all output must be taken into account, each column must sum to 1. This fact helps to fill in any missing entries.

<table>
<thead>
<tr>
<th>Distribution of Output from:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
</tr>
<tr>
<td>.65</td>
</tr>
<tr>
<td>.05</td>
</tr>
<tr>
<td>.30</td>
</tr>
</tbody>
</table>

2. Since $x_5 \leq 500$, the equations D and A for $x_1$ and $x_2$ imply that $x_1 \geq 100$ and $x_2 \leq 700$. The fact that $x_5 \geq 0$ implies that $x_1 \leq 600$ and $x_2 \geq 200$. So, $100 \leq x_1 \leq 600$, and $200 \leq x_2 \leq 700$.

### 1.7 LINEAR INDEPENDENCE

The homogeneous equations in Section 1.5 can be studied from a different perspective by writing them as vector equations. In this way, the focus shifts from the unknown solutions of $Ax = 0$ to the vectors that appear in the vector equations.
For instance, consider the equation
\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} x_1 + 
\begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix} x_2 + 
\begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} x_3 = 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
(1)

This equation has a trivial solution, of course, where \( x_1 = x_2 = x_3 = 0 \). As in Section 1.5, the main issue is whether the trivial solution is the only one.

**Definition**

An indexed set of vectors \( \{v_1, \ldots, v_p\} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation
\[
x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0
\]
has only the trivial solution. The set \( \{v_1, \ldots, v_p\} \) is said to be **linearly dependent** if there exist weights \( c_1, \ldots, c_p \), not all zero, such that
\[
c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0
\]
(2)

Equation (2) is called a **linear dependence relation** among \( v_1, \ldots, v_p \) when the weights are not all zero. An indexed set is linearly dependent if and only if it is not linearly independent. For brevity, we may say that \( v_1, \ldots, v_p \) are linearly dependent when we mean that \( \{v_1, \ldots, v_p\} \) is a linearly dependent set. We use analogous terminology for linearly independent sets.

**Example 1**

Let \( v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), \( v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \), and \( v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \).

a. Determine if the set \( \{v_1, v_2, v_3\} \) is linearly independent.

b. If possible, find a linear dependence relation among \( v_1, v_2, \) and \( v_3 \).

**SOLUTION**

a. We must determine if there is a nontrivial solution of equation (1) above. Row operations on the associated augmented matrix show that
\[
\begin{bmatrix}
1 & 4 & 2 & 0 \\
2 & 5 & 1 & 0 \\
3 & 6 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 4 & 2 & 0 \\
0 & -3 & -3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Clearly, \( x_1 \) and \( x_2 \) are basic variables, and \( x_3 \) is free. Each nonzero value of \( x_3 \) determines a nontrivial solution of (1). Hence \( v_1, v_2, v_3 \) are linearly dependent (and not linearly independent).

b. To find a linear dependence relation among \( v_1, v_2, \) and \( v_3 \), completely row reduce the augmented matrix and write the new system:
\[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \begin{aligned}
x_1 &- 2x_3 = 0 \\
x_2 &+ x_3 = 0 \\
& 0 = 0
\end{aligned}
\]

Thus \( x_1 = 2x_3, x_2 = -x_3, \) and \( x_3 \) is free. Choose any nonzero value for \( x_3 \)—say, \( x_3 = 5 \). Then \( x_1 = 10 \) and \( x_2 = -5 \). Substitute these values into equation (1) and obtain
\[
10v_1 - 5v_2 + 5v_3 = 0
\]

This is one (out of infinitely many) possible linear dependence relations among \( v_1, v_2, \) and \( v_3 \).
Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [a_1 \cdots a_n]$ instead of a set of vectors. The matrix equation $Ax = \mathbf{0}$ can be written as

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = \mathbf{0}$$

*Each linear dependence relation among the columns of $A$ corresponds to a nontrivial solution of $Ax = \mathbf{0}$. Thus we have the following important fact.*

The columns of a matrix $A$ are linearly independent if and only if the equation $Ax = \mathbf{0}$ has only the trivial solution. \hfill (3)

**EXAMPLE 2** Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$ are linearly independent.

**SOLUTION** To study $Ax = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

At this point, it is clear that there are three basic variables and no free variables. So the equation $Ax = \mathbf{0}$ has only the trivial solution, and the columns of $A$ are linearly independent.

**Sets of One or Two Vectors**

A set containing only one vector—say, $v$—is linearly independent if and only if $v$ is not the zero vector. This is because the vector equation $x_1v = \mathbf{0}$ has only the trivial solution when $v \neq \mathbf{0}$. The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

The next example will explain the nature of a linearly dependent set of two vectors.

**EXAMPLE 3** Determine if the following sets of vectors are linearly independent.

a. $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$  

b. $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

**SOLUTION**

a. Notice that $v_2$ is a multiple of $v_1$, namely, $v_2 = 2v_1$. Hence $-2v_1 + v_2 = \mathbf{0}$, which shows that $\{v_1, v_2\}$ is linearly dependent.

b. The vectors $v_1$ and $v_2$ are certainly not multiples of one another. Could they be linearly dependent? Suppose $c$ and $d$ satisfy

$$cv_1 + dv_2 = \mathbf{0}$$

If $c \neq 0$, then we can solve for $v_1$ in terms of $v_2$, namely, $v_1 = (-d/c)v_2$. This result is impossible because $v_1$ is not a multiple of $v_2$. So $c$ must be zero. Similarly, $d$ must also be zero. Thus $\{v_1, v_2\}$ is a linearly independent set.
The arguments in Example 3 show that you can always decide by inspection when a set of two vectors is linearly dependent. Row operations are unnecessary. Simply check whether at least one of the vectors is a scalar times the other. (The test applies only to sets of two vectors.)

A set of two vectors \{v_1, v_2\} is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin. Figure 1 shows the vectors from Example 3.

**Sets of Two or More Vectors**

The proof of the next theorem is similar to the solution of Example 3. Details are given at the end of this section.

**Theorem 7**  
Characterization of Linearly Dependent Sets
An indexed set \(S = \{v_1, \ldots, v_p\}\) of two or more vectors is linearly dependent if and only if at least one of the vectors in \(S\) is a linear combination of the others. In fact, if \(S\) is linearly dependent and \(v_1 \neq \mathbf{0}\), then some \(v_j\) (with \(j > 1\)) is a linear combination of the preceding vectors, \(v_1, \ldots, v_{j-1}\).

**Warning:** Theorem 7 does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors. See Practice Problem 3.

**Example 4**  
Let \(u = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\) and \(v = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}\). Describe the set spanned by \(u\) and \(v\), and explain why a vector \(w\) is in \(\text{Span}\{u, v\}\) if and only if \(\{u, v, w\}\) is linearly dependent.

**Solution** The vectors \(u\) and \(v\) are linearly independent because neither vector is a multiple of the other, and so they span a plane in \(\mathbb{R}^3\). (See Section 1.3.) In fact, \(\text{Span}\{u, v\}\) is the \(x_1x_2\)-plane (with \(x_3 = 0\)). If \(w\) is a linear combination of \(u\) and \(v\), then \(\{u, v, w\}\) is linearly dependent, by Theorem 7. Conversely, suppose that \(\{u, v, w\}\) is linearly dependent. By Theorem 7, some vector in \(\{u, v, w\}\) is a linear combination of the preceding vectors (since \(u \neq \mathbf{0}\)). That vector must be \(w\), since \(v\) is not a multiple of \(u\). So \(w\) is in \(\text{Span}\{u, v\}\). See Fig. 2.
Example 4 generalizes to any set \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \) in \( \mathbb{R}^3 \) with \( \mathbf{u} \) and \( \mathbf{v} \) linearly independent. The set \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \) will be linearly dependent if and only if \( \mathbf{w} \) is in the plane spanned by \( \mathbf{u} \) and \( \mathbf{v} \).

The next two theorems describe special cases in which the linear dependence of a set is automatic. Moreover, Theorem 8 will be a key result for work in later chapters.

**THEOREM 8**

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w} \} \) in \( \mathbb{R}^n \) is linearly dependent if \( p > n \).

**PROOF** Let \( A = [ \mathbf{v}_1 \cdots \mathbf{v}_p ] \). Then \( A \) is \( n \times p \), and the equation \( A \mathbf{x} = \mathbf{0} \) corresponds to a system of \( n \) equations in \( p \) unknowns. If \( p > n \), there are more variables than equations, so there must be a free variable. Hence \( A \mathbf{x} = \mathbf{0} \) has a nontrivial solution, and the columns of \( A \) are linearly dependent. See Fig. 3 for a matrix version of this theorem.

**Warning:** Theorem 8 says nothing about the case in which the number of vectors in the set does not exceed the number of entries in each vector.

**EXAMPLE 5** The vectors \( \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \) are linearly dependent by Theorem 8, because there are three vectors in the set and there are only two entries in each vector. Notice, however, that none of the vectors is a multiple of one of the other vectors. See Fig. 4.

**THEOREM 9**

If a set \( S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) in \( \mathbb{R}^n \) contains the zero vector, then the set is linearly dependent.

**PROOF** By renumbering the vectors, we may suppose \( \mathbf{v}_1 = \mathbf{0} \). Then the equation \( 1 \mathbf{v}_1 + 0 \mathbf{v}_2 + \cdots + 0 \mathbf{v}_p = \mathbf{0} \) shows that \( S \) is linearly dependent.

**EXAMPLE 6** Determine by inspection if the given set is linearly dependent.

a. \( \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \)

b. \( \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} \)

c. \( \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 10 \end{bmatrix}, \begin{bmatrix} -9 \\ 15 \end{bmatrix} \)

**SOLUTION**

a. The set contains four vectors, each of which has only three entries. So the set is linearly dependent by Theorem 8.

b. Theorem 8 does not apply here because the number of vectors does not exceed the number of entries in each vector. Since the zero vector is in the set, the set is linearly dependent by Theorem 9.

c. Compare the corresponding entries of the two vectors. The second vector seems to be \(-3/2\) times the first vector. This relation holds for the first three pairs of entries, but fails for the fourth pair. Thus neither of the vectors is a multiple of the other, and hence they are linearly independent.
In general, you should read a section thoroughly several times to absorb an important concept such as linear independence. The notes in the Study Guide for this section will help you learn to form mental images of key ideas in linear algebra. For instance, the following proof is worth reading carefully because it shows how the definition of linear independence can be used.

PROOF OF THEOREM 7 (Characterization of Linearly Dependent Sets)

If some \( \mathbf{v}_j \) in \( S \) equals a linear combination of the other vectors, then \( \mathbf{v}_j \) can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight \((-1)\) on \( \mathbf{v}_j \). [For instance, if \( \mathbf{v}_1 = c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \), then \( \mathbf{0} = (-1)\mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + 0 \mathbf{v}_4 + \cdots + 0 \mathbf{v}_p ) \) Thus \( S \) is linearly dependent.

Conversely, suppose \( S \) is linearly dependent. If \( \mathbf{v}_1 \) is zero, then it is a (trivial) linear combination of the other vectors in \( S \). Otherwise, \( \mathbf{v}_1 \neq \mathbf{0} \), and there exist weights \( c_1, \ldots, c_p \), not all zero, such that

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p = \mathbf{0}
\]

Let \( j \) be the largest subscript for which \( c_j \neq 0 \). If \( j = 1 \), then \( c_1 \mathbf{v}_1 = \mathbf{0} \), which is impossible because \( \mathbf{v}_1 \neq \mathbf{0} \). So \( j > 1 \), and

\[
c_1 \mathbf{v}_1 + \cdots + c_j \mathbf{v}_j + 0 \mathbf{v}_{j+1} + \cdots + 0 \mathbf{v}_p = \mathbf{0}
\]

\[
c_j \mathbf{v}_j = -c_1 \mathbf{v}_1 - \cdots - c_{j-1} \mathbf{v}_{j-1}
\]

\[
\mathbf{v}_j = \left( -\frac{c_1}{c_j} \right) \mathbf{v}_1 + \cdots + \left( -\frac{c_{j-1}}{c_j} \right) \mathbf{v}_{j-1}
\]

PRACTICE PROBLEMS

Let \( \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \), \( \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix} \), \( \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \), and \( \mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix} \).

1. Are the sets \{\mathbf{u}, \mathbf{v}\}, \{\mathbf{u}, \mathbf{w}\}, \{\mathbf{u}, \mathbf{z}\}, \{\mathbf{v}, \mathbf{w}\}, \{\mathbf{v}, \mathbf{z}\}, \text{ and } \{\mathbf{w}, \mathbf{z}\} \) each linearly independent? Why or why not?

2. Does the answer to Problem 1 imply that \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\} \) is linearly independent?

3. To determine if \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\} \) is linearly dependent, is it wise to check if, say, \( \mathbf{w} \) is a linear combination of \( \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{z} \)?

4. Is \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\} \) linearly dependent?

1.7 EXERCISES

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

1. \[
\begin{bmatrix}
  5 \\
  0 \\
  0
\end{bmatrix}, \begin{bmatrix}
  7 \\
  2 \\
  -6
\end{bmatrix}, \begin{bmatrix}
  9 \\
  4 \\
  -8
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
  0 \\
  2 \\
  3
\end{bmatrix}, \begin{bmatrix}
  0 \\
  0 \\
  -8
\end{matrix}, \begin{bmatrix}
  -1 \\
  3 \\
  1
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
  2 \\
  -3 \\
  -4
\end{bmatrix}, \begin{bmatrix}
  6 \\
  -8 \\
  -9
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
  3 \\
  -1 \\
  3
\end{bmatrix}, \begin{bmatrix}
  9 \\
  -3 \\
  -9
\end{bmatrix}
\]

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

5. \[
\begin{bmatrix}
  0 & -3 & 9 \\
  2 & 1 & -7 \\
 -1 & 4 & -5 \\
 1 & -4 & -2
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
  -4 & -3 & 0 \\
  0 & -1 & 5 \\
  1 & 1 & -5 \\
  2 & 1 & -10
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
  1 & 4 & -3 & 0 \\
 -2 & 7 & 5 & 1 \\
 -4 & -5 & 7 & 5
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
  1 & -2 & 3 & 2 \\
 -2 & 4 & -6 & 2 \\
 0 & 1 & -1 & 3
\end{bmatrix}
\]

In Exercises 9 and 10, (a) for what values of \( h \) is \( \mathbf{v}_3 \) in Span \{\mathbf{v}_1, \mathbf{v}_2\}, and (b) for what values of \( h \) is \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) linearly dependent? Justify each answer.
9. \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix} \)

10. \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ 15 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix} \)

In Exercises 11–14, find the value(s) of \( h \) for which the vectors are linearly dependent. Justify each answer.

11. \( \begin{bmatrix} 2 \\ -2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 7 \\ h \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ -3 \\ 9 \end{bmatrix} \)

12. \( \begin{bmatrix} 3 \\ 4 \\ -6 \\ h \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ -4 \\ 9 \end{bmatrix} \)

13. \( \begin{bmatrix} 1 \\ 5 \\ -9 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \\ -9 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} \)

14. \( \begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 2 \\ 6 \end{bmatrix} \)

Determine by inspection whether the vectors in Exercises 15–20 are linearly independent. Justify each answer.

15. \( \begin{bmatrix} 5 \\ 2 \\ 1 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} \)

16. \( \begin{bmatrix} 2 \\ -4 \\ 8 \\ 6 \\ -12 \end{bmatrix} \)

17. \( \begin{bmatrix} 5 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -7 \end{bmatrix} \)

18. \( \begin{bmatrix} 2 \\ -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ 0 \\ 0 \end{bmatrix} \)

19. \( \begin{bmatrix} -8 \\ -12 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -7 \\ 3 \\ 0 \end{bmatrix} \)

20. \( \begin{bmatrix} 1 \\ 4 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \)

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of the material read in the text.

21. a. The columns of a matrix \( A \) are linearly independent if the equation \( Ax = 0 \) has the trivial solution.
   b. If \( S \) is a linearly dependent set, then each vector in \( S \) is a linear combination of the other vectors in \( S \).
   c. The columns of any \( 4 \times 5 \) matrix are linearly dependent.
   d. If \( x \) and \( y \) are linearly independent, and if \( \{x, y, z\} \) is linearly dependent, then \( z \) is in \( \text{Span} \{x, y\} \).

22. a. If \( u \) and \( v \) are linearly independent, and if \( w \) is in \( \text{Span} \{u, v\} \), then \( \{u, v, w\} \) is linearly independent.
   b. If three vectors in \( \mathbb{R}^3 \) lie in the same plane in \( \mathbb{R}^3 \), then they are linearly dependent.
   c. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.
   d. If a set in \( \mathbb{R}^n \) is linearly dependent, then the set contains more than \( n \) vectors.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23. \( A \) is a \( 2 \times 2 \) matrix with linearly dependent columns.

24. \( A \) is a \( 3 \times 3 \) matrix with linearly independent columns.

25. \( A \) is a \( 4 \times 2 \) matrix, \( A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \), and \( a_2 \) is not a multiple of \( a_1 \).

26. \( A \) is a \( 4 \times 3 \) matrix, \( A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \), such that \( \{a_1, a_2\} \) is linearly independent and \( a_3 \) is not in \( \text{Span} \{a_1, a_2\} \).

27. How many pivot columns must a \( 6 \times 4 \) matrix have if its columns are linearly independent? Why?

28. How many pivot columns must a \( 4 \times 6 \) matrix have if its columns span \( \mathbb{R}^4 \)? Why?

29. Construct \( 3 \times 2 \) matrices \( A \) and \( B \) such that \( Ax = 0 \) has a nontrivial solution, but \( Bx = 0 \) has only the trivial solution.

30. a. Fill in the blank in the following statement: “If \( A \) is an \( m \times n \) matrix, then the columns of \( A \) are linearly independent if and only if \( A \) has _____ pivot columns.”
   b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved without performing row operations. [Hint: Write \( Ax = 0 \) as a vector equation.]

31. Given \( A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix} \)
   observe that the third column is the sum of the first two columns. Find a nontrivial solution of \( Ax = 0 \).

32. Given \( A = \begin{bmatrix} -4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix} \)
   observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of \( Ax = 0 \).

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a counterexample to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

33. If \( v_1, \ldots, v_4 \) are in \( \mathbb{R}^4 \) and \( v_1 = 2v_1 + v_2 + v_3 \), then \( \{v_1, v_2, v_3, v_4\} \) is linearly dependent.

34. If \( v_1 \) and \( v_2 \) are in \( \mathbb{R}^4 \) and \( v_2 \) is not a scalar multiple of \( v_1 \), then \( \{v_1, v_2\} \) is linearly independent.

35. If \( v_1, \ldots, v_3 \) are in \( \mathbb{R}^5 \) and \( v_3 = 0 \), then \( \{v_1, v_2, v_3, v_4, v_5\} \) is linearly dependent.

36. If \( v_1, v_2, v_3 \) are in \( \mathbb{R}^3 \) and \( v_3 \) is not a linear combination of \( v_1, v_2 \), then \( \{v_1, v_2, v_3\} \) is linearly independent.

37. If \( v_1, \ldots, v_4 \) are in \( \mathbb{R}^4 \) and \( \{v_1, v_2, v_3\} \) is linearly dependent, then \( \{v_1, v_2, v_3, v_4\} \) is also linearly dependent.

38. If \( \{v_1, \ldots, v_4\} \) is a linearly independent set of vectors in \( \mathbb{R}^4 \), then \( \{v_1, v_2, v_3\} \) is also linearly independent. [Hint: Think about \( x_1v_1 + x_2v_2 + x_3v_3 + 0 \cdot v_4 = 0 \).]

39. Suppose \( A \) is an \( m \times n \) matrix with the property that for all \( b \) in \( \mathbb{R}^m \) the equation \( Ax = b \) has at most one solution. Use the
definition of linear independence to explain why the columns of $A$ must be linearly independent.

40. Suppose an $m \times n$ matrix $A$ has $n$ pivot columns. Explain why for each $b$ in $\mathbb{R}^m$ the equation $Ax = b$ has at most one solution. [Hint: Explain why $Ax = b$ cannot have infinitely many solutions.]

[M] In Exercises 41 and 42, use as many columns of $A$ as possible to construct a matrix $B$ with the property that the equation $Bx = 0$ has only the trivial solution. Solve $Bx = 0$ to verify your work.

41. $A = \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ -5 & -3 & -7 & -11 & 15 \\ 8 & -7 & 23 & 4 & 15 \end{bmatrix}$

42. $A = \begin{bmatrix} 12 & 10 & -6 & 8 & 4 & -14 \\ -7 & -6 & 4 & -5 & -7 & 9 \\ 9 & 9 & -9 & 9 & -9 & -18 \\ -4 & -3 & -1 & 0 & -8 & 1 \\ 8 & 7 & -5 & 6 & 1 & -11 \end{bmatrix}$

43. [M] With $A$ and $B$ as in Exercise 41, select a column $v$ of $A$ that was not used in the construction of $B$ and determine if $v$ is in the set spanned by the columns of $B$. (Describe your calculations.)

44. [M] Repeat Exercise 43 with the matrices $A$ and $B$ from Exercise 42. Then give an explanation for what you discover, assuming that $B$ was constructed as specified.

**Solutions to Practice Problems**

1. Yes. In each case, neither vector is a multiple of the other. Thus each set is linearly independent.

2. No. The observation in Practice Problem 1, by itself, says nothing about the linear independence of $\{u, v, w, z\}$.

3. No. When testing for linear independence, it is usually a poor idea to check if one selected vector is a linear combination of the others. It may happen that the selected vector is not a linear combination of the others and yet the whole set of vectors is linearly dependent. In this practice problem, $w$ is not a linear combination of $u, v,$ and $z$.

4. Yes, by Theorem 8. There are more vectors (four) than entries (three) in them.

### 1.8 Introduction to Linear Transformations

The difference between a matrix equation $Ax = b$ and the associated vector equation $x_1a_1 + \cdots + x_na_n = b$ is merely a matter of notation. However, a matrix equation $Ax = b$ can arise in linear algebra (and in applications such as computer graphics and signal processing) in a way that is not directly connected with linear combinations of vectors. This happens when we think of the matrix $A$ as an object that “acts” on a vector $x$ by multiplication to produce a new vector called $Ax$.

For instance, the equations

\[
\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

say that multiplication by $A$ transforms $x$ into $b$ and transforms $u$ into the zero vector. See Fig. 1.